# QUADRATIC SUPERALGEBRAS IN MATHEMATICS AND PHYSICS

by

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# ABSTRACT

We introduce in this thesis a class of quadratic deformations of Lie superalgebras which we term *quadratic superalgebras*. These are finitely-generated algebras with a $\mathbb{Z}_2$-graded structure comprising an even and an odd part; the even part is an ordinary Lie algebra, the odd part is a module of the even part, and the anticommutator of two odd elements closes quadratically on the even generators. One motivation to study algebras with these structural properties is their arising in the observable algebra of gauge invariant fields in Hamiltonian lattice QCD [44] and the subsequent study of polynomial $gl(n)$ superalgebras [45]. The present work both broadens the scope and extends the analysis of the latter. For this class of algebras we derive a Poincaré-Birkhoff-Witt theorem, including an explicit ordered basis, which we then employ as a means to investigate the structure of irreducible modules; these are analogous to Kac modules for Lie superalgebras. Further rationale to study quadratic superalgebras is due to the remarkable existence of zero-step modules; these are so-called atypical modules for which the entire irreducible module of the quadratic superalgebra consists of a single irreducible module of the even subalgebra.

In addition to their mathematical aspects, we investigate in this thesis an application of quadratic superalgebras in the context of space-time conformal supersymmetry. We show that the algebra of $\mathcal{N} = 1$ space-time conformal supersymmetry, $su(2, 2/1)$, arises as a contraction limit of a certain quadratic superalgebra. In this setting we exploit the existence of zero-step modules which, for a fixed parameter choice of the quadratic family under consideration, coincide with the massless positive energy unitary irreducible representations (in the standard classification of Mack) of the even subalgebra. For these massless particle multiplets the odd generators vanish identically and supersymmetry is carried (unbroken) without the accompaniment of superpartners. Thus, in the context of extended non-linear symmetry principles and their role in determining the spectrum of fundamental particles, we point out that there exist candidate algebraic structures which implement (extended) supersymmetric invariance while at the same time obviating the need for every particle of the standard model to be accompanied by a superpartner.

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# TABLE OF CONTENTS

# LIST OF TABLES

Chapter 1

# INTRODUCTION

The application of symmetry principles together with the investigation of the correct mathematical structures in which to frame the laws of physics are central themes in theoretical physics. The mathematical expression of physical symmetries typically describes a group structure and until recently the dominant structures have been those of the discrete groups and Lie groups and their associated algebras. The classification of Lie algebras was completed by Cartan in 1894 and Noether's theorem [59], proven in 1915, provided a direct connection between physical symmetries and conserved quantities.

Aside from elucidating the connection between the global symmetries of rotation, translation and time and the corresponding conservation of angular momentum, linear momentum and (mass-)energy respectively, early successes of the group theoretical approach were made in quantum mechanics beginning in the 1920's [11]. Significant contributions included both discrete symmetries such as permutation, parity, charge conjugation and time reversal (due largely to the work of Heisenberg, Weyl, Wigner and Dirac) along with gauge symmetries $U(1)$ and later $U(2)$ for electromagnetism and isospin culminating in the work of Yang and Mills in 1954 [79].

The application of Lie groups to particle physics expanded greatly in the 1960's where attempts were being made to develop a theory that unified internal symmetries with relativistic invariance. The scope of these investigations was ultimately restricted in 1967 by the Coleman-Mandula *no-go* theorem which limited candidate models to those equivalent to a tensor product of the Poincaré group with an internal symmetry [18]. The development of the standard model, a spontaneously broken nonabelian Yang-Mills Higgs gauge theory, was complete by the mid 1970's with internal symmetries $SU(3) \times SU(2) \times SU(1)$ which, in addition to the fermionic and electroweak fields, included the prediction of the Higgs field.

Despite its many successes, and ongoing experimental confirmation in high energy particle physics, the standard model has significant limitations. These include the failure to explain dark matter or the matter-antimatter asymmetry of the universe and, of course, the omission of gravity [5]. It was in this context that supersymmetry emerged in the early 1970's, launched initially by Gel'fand and Likhtman followed soon after by the influential work of Wess and Zumino [74]. Supersymmetry is a symmetry between bosons and fermions placing within a single algebraic framework a model of particles *and* their interactions. As an extension of the standard model, supersymmetry offers potentially elegant solutions to many of the shortcomings of the standard model leading to the possibility of a unified model in its gauge invariant form known as supergravity [30, 22].

At the technical level supersymmetry is handled by Lie superalgebras for which the classification and introductory aspects of the representation theory were completed by Kac in 1977 [48, 49]. Candidate models of supersymmetry are classified broadly by $\mathcal{N} = 1, 2, 4, 8$ characterising the number of real spinor representations comprising the fermionic part and $\mathcal{D}$ corresponding to the physical dimension, $\mathcal{D} = 4$ being standard spacetime. $\mathcal{N} = 1$ is the so-called *minimal* model containing the smallest possible number of odd generators and $\mathcal{N} > 1$ models are called extended supersymmetries. In 1975 the Haag−Lopuszański−Sohnius (HLS) theorem confirmed the validity of these models, the theorem serving as a supersymmetric generalisation of the Coleman-Mandula Theorem under mildly weakened assumptions [40]. Despite its successes as a theoretical tool, the ultimate shortcoming of all supersymmetric theories is the lack of experimental evidence to date. In spite of ever increasing energy thresholds in experimental particle physics, up to and including the first run of the Large Hadron Collider, predicted superparticles of the known particles in the standard model have not appeared pushing upwards the bounds on superparticle masses and significantly constraining candidate models [5].

The reach of the so-called no-go theorems is limited to models of exact or unbroken symmetries. The breaking of supersymmetry is in fact taken for granted as the translational invariance of the odd generators, as required by the HLS theorem, would otherwise constrain superpartners to have the same mass as their elementary counterparts [68]. The occurrence of symmetry breaking in relation to energy scales means that low energy states may not respect supersymmetry, offering the potential to lift the mass degeneracy, whereas higher energy states would continue to manifest or 'reinstate' the symmetry exactly.

The imperative of symmetry breaking may also be framed more algebraically, with the possibility of extended or non-standard (super)symmetries being respected asymptotically, which in the appropriate energy limit contract to a standard model of supersymmetry, that is, a model respecting the constraints of the HLS theorem [76]. Candidates for such algebraic structures might include nonlinear extensions or deformations of the classical symmetry groups. Indeed, simple examples of non-linear symmetries (in this case concerning finite $\mathcal{W}$-algebras - discussed below) have been shown to arise in elementary systems; these include well-known dynamical systems that posses a Coulomb potential and the two-dimensional anisotropic harmonic oscillator [20].

The deformation of an algebra is the definition of a larger parametrised family of more general algebraic structures for which the original is recovered for a certain parameter value. For example the Lorentz group may be viewed as a deformation of the Galilean group in terms of the parameter $\frac{1}{c}$ and the so-called deformation quantisation of classical mechanics provides an alternative route to quantum mechanics with $\hbar$ viewed as the deformation parameter [4].

Quantum groups and associated $q$-deformed algebras are perhaps the most common example of a deformation structure. These exist for the enveloping algebras of ordinary Lie algebras with the non-deformed algebra recovered in the limit $q \to 1$. Applications of quantum groups emerged in the early 1980's in the context of integrable models in statistical mechanics via the Yang-Baxter equations and quantum inverse scattering techniques [72, 52].

In 1985 Drinfeld [23] framed quantum groups within the structure of Hopf algebras which, building upon the work of Jimbo [47], placed the theory within a robust mathematical framework. Further applications of quantum groups in the context of gauge theories include: $q$-deformations of the electroweak gauge group $SU(2) \times U(1)$ [17], a generalisation of the hydrogen atom system from $so(4)$ to $su_q(2) \otimes su_q(2)$ [50] and the notion of $q$-gravity [25].

Related to quantum groups are the nonlinear algebras known as Yangians and $\mathcal{W}$-algebras. The former is in fact a family of quantum groups related to solutions of the $R$-matrix and Yang-Baxter equations, while the latter arose within the field of $2D$ conformal field theory finding applications in string theory [8]. There exist some remarkable connections between Yangians and the class of so-called finite $\mathcal{W}$-algebras [64, 13]; finite $\mathcal{W}$-algebras will also be discussed, in light of the present work, at the conclusion of this thesis.

In this algebraic context, non-linearity refers to the defining (multiplicative) relations between elements of a generating set. $\mathcal{W}$-algebras and quantum groups admit defining relations ranging from a maximal (polynomial) degree of 2 in the case of finite $\mathcal{W}$-algebras, and up to an infinite series of (formal) expansion terms for quantum groups more generally. An alternative approach for exploring algebraic structures that go beyond linear groups is to limit the degree of non-linearity to quadratic terms, relaxing the requirement of an underlying co-algebra structure and demanding instead that the algebra possess the cohomologically defined property of being Koszul [62]. Algebras defined within this framework are broadly known as *quadratic algebras* and include as a subclass the so-called *Poincaré-Birkhoff-Witt algebras* (PBW) which satisfy an analogue of the classical PBW theorem for Lie algebras; for a detailed presentation of the topic we refer the reader to the textbook of Polishchuk and Positselski [61].

Ordinary Lie (super)algebras are of course examples of PBW algebras for which the PBW theorem may be expressed as an isomorphism between the associated graded algebra of the universal enveloping algebra and the symmetric algebra. The enveloping algebra acquires a basis of ordered monomials from the symmetric algebra, subject only to fulfillment of the Jacobi identities. The key idea behind the generalised PBW theorem is recognising that the symmetric algebra may be viewed as the tensor algebra of the underlying vector space factored by the ideal generated by the quadratic projection (also called the homogeneous part) of the defining relations [12]. It turns out, in the context of Koszul algebras, that the necessary conditions for this isomorphism to hold more generally are the fulfillment of a certain set of cubic identities that generalise the ordinary Jacobi identities.

In practice, the role of the PBW theorem is to provide an ordered basis for the algebra in question, thus enabling the standard tools of the representation theory such as the implementation of weight space techniques. In relation to symmetry principles, the representation theory of the associated symmetry group gives information about the physical states of the system; for example the classification of fundamental particles states is determined via the study of unitary irreducible representations. For this reason the PBW property is a highly desirable characteristic of any candidate for a non-linear symmetry group.

Certain classes of non-linear algebras have been shown to satisfy an analogue of the classical PBW theorem. Proof of the theorem for Yangians and quantum groups can be found in several publications including [23, 55] and later [58]. Similarly for finite $\mathcal{W}$-algebras [20] and affine $\mathcal{W}$-algebras [70]. In the context of two-dimensional conformal field theory

De Sole and Kac [69] have introduced a class of non-linear superalgebras defined in terms of a so-called $\lambda$-bracket; these generalise the notion of a vertex algebra and satisfy a PBW theorem on very general grounds (see also [24]). Fradkin and Linetsky classify a category of $D = 2$ operator product expansion algebras with quadratic non-linearity which are shown to satisfy Jacobi Identities [27]. For further examples we refer the reader to the broad review of PBW algebras by Shepler and Witherspoon [66].

In this thesis we introduce a class of quadratic deformations of the universal enveloping algebra of ordinary Lie superalgebras. Given an underlying $\mathbb{Z}_2$-graded vector space $L = L_{\bar{0}} \oplus L_{\bar{1}}$, the defining relations satisfy non-linear graded commutation relations of the form

$$[L_{\bar{0}}, L_{\bar{0}}] \subset L_{\bar{0}} \qquad [L_{\bar{0}}, L_{\bar{1}}] \subset L_{\bar{1}} \qquad \{L_{\bar{1}}, L_{\bar{1}}\} \subset (L_{\bar{0}} \otimes L_{\bar{0}}) + L_{\bar{0}} + \mathbb{C}. \tag{1.1}$$

We derive constraints on the structure constants consistent with a set of generalised Jacobi identities and prove an analogue of the PBW theorem including an ordered monomial basis. We also explore an application of these algebras in the context of spacetime conformal supersymmetry where we exploit the remarkable existence of zero-step modules - a detailed review of the thesis content is given in the thesis outline below.

The present work takes inspiration from the study of gauge invariant fields in Hamiltonian QCD lattice [44] and builds upon the initial analysis of $gl(n)$-type polynomial superalgebras [45]. In the context of Hamiltonian QCD lattice, the observable algebra comprises colour invariant operators built from quark fields taking the form

$$\psi_{ijk} = \varepsilon^{abc} f_{ai} f_{bj} f_{ck} \qquad \text{and} \qquad \overline{\psi}^{ijk} = \varepsilon^{abc} f^{\dagger}_{ai} f^{\dagger}_{bj} f^{\dagger}_{ck}, \tag{1.2}$$

where $i, j, k = 1, 2, ..., n$ are spatial lattice, spin and flavour labels, $a, b, c = 1, 2, 3$ are colour labels and $f$, $f^{\dagger}$ are canonical fermionic creation and annihilation operators. It is easily verified that these (odd) operators satisfy quadratic anticommutation relations, which together with the (even) gluonic field invariants $E^{i}_{a,j} = f^{\dagger}_{ai} f_{aj}$ comprise a $\mathbb{Z}_2$-graded set of generators satisfying the (quadratic) structural relations (1.1).

The study of polynomial superalgebras by Jarvis and Rudolph [45] is an important antecedent to our work, where in the restricted setting $L_{\bar{0}} = gl(n)$, a class of $\mathbb{Z}_2$-graded algebras satisfying (1.1) were investigated. Within this class, a one-parameter family of algebras was discovered via the derivation of admissible structure constants consistent with the ordinary (graded) Jacobi identities in their nested bracket form. Some limitations of the analysis in [45] are: the restriction of the even subalgebra to $gl(n)$, the absence of a PBW theorem and associated results concerning the representation theory, and the implications of having abstractly imposed certain additional quadratic relations that originate in oscillator-type realisations. The latter turn out to have consequences for the existence of a PBW basis and the parametrisation of the associated family of algebras.

### Thesis Outline

The thesis begins with two chapters of mathematical background. The first gives a review of the theory of Lie algebras and Lie superalgebras including definitions, key results and an overview of their classification. The review, whilst lacking the rigor to be considered pedagogical, aims to present the topics and worked examples with a sufficient degree of detail to serve as a reference point for notational conventions, canonical presentations and established results. The second chapter concerns the theory of real forms and characteristic identities of Lie algebras. The results and techniques of the latter, due largely to work of Gould [35], have a significant role to play in the structural relations of quadratic superalgebras.

Chapter 4 introduces the notion of *quadratic superalgebras* which, as a broad class of algebras, are the central object around which the original work of this thesis is based. This technical chapter includes key definitions together with the presentation of the important structural results such as the form of the structure constants and Jacobi identities, the PBW theorem and derivation of an ordered basis, as well as initial steps towards investigating the structure of induced modules, including those which are atypical. We also present certain of these results within the framework of *quadratic algebras*. This provides a powerful toolkit through which to analyse possible generalisations of quadratic algebras and in particular the conditions necessary for the existence of a PBW basis of ordered monomials.

In chapter 5 we turn to a specific family of quadratic superalgebras which we denote $gl_2(n/1)$. This one-parameter family constitutes the most general solution possible for a quadratic superalgebra with even part $L_{\bar{0}} = gl(n)$. We provide in this chapter a derivation of the structure constants (reiterating [45]), proof of the degeneracy of the $n = 2$ case to $sl(2/1)$ and an initial investigation into the atypicality conditions for certain classes of representations. We show that the ordinary Lie superalgebra $sl(n/1)$ is obtained in a certain contraction limit of the free parameter. Finally, in §5.4 further generalisations are considered; these include an enlargement of the class $gl_2(n/1)$ via the imposition of additional structural relations and a mild generalisation of the action of the even subalgebra on the odd generators.

The final technical chapter is devoted to an application of $gl_2(n/1)$ in the context of spacetime supersymmtery. Building upon the ideas nascent in [43] we impose the real form $gl_2(2,2/1)$ of the $n = 4$ case as a nonlinear extension of $su(2,2/1)$ the algebra of $\mathcal{N} = 1$ space-time conformal supersymmetry. We review the classification due to Mack [56] of positive energy unitary representations of the subalgebra $su(2,2)$, focusing in particular on the class of conformally invariant massless representations. Using the techniques of characteristic identities, we prove that the even part $u(2,2)$ satisfies a minimal identity that is quadratic for precisely the class of massless representations. Finally, we show that the quadratic anticommutator of $gl_2(2,2/1)$ may be brought into correspondence with this identity thus identifying the massless representations as zero-step modules. Consequently, the supersymmetry generators are identically zero and the quadratic superalgebra is carried entirely on a single (irreducible) multiplet of the even subalgebra, thus obviating the need for supersymmetric partners.

The thesis is concluded in chapter 7 where the key results are reiterated in the context of related literature and topics for future work. Also included is an appendix; §A.1 identifies isomorphisms between certain instances of $gl_2(n/1)$, §A.2 provides a new (algebraic) derivation of Mack's conditions for masslessness in the context of positive energy unitary representation of the conformal group, and finally §A.2 gives a brief overview of the construction of oscillator representation for $su(2,2)$.

Contained in the introduction of each of the chapters 4, 5 and 6 is an overview of the original results due to the candidate. Clarification of authorship is provided in the text where a co-author has been the primary contributor of a specific result that has been published in one of the jointly-prepared papers [46, 82].

A list of symbols (nomenclature) may be found on page 90. The Einstein summation convention for repeated indices is used throughout this thesis unless otherwise indicated. The symbol ${\delta^i}_j$ is the Kronecker delta satisfying

$${\delta^i}_j = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases}$$

## Chapter 2

# LIE ALGEBRAS AND LIE SUPERALGEBRAS

This chapter contains key definitions and well-known results as concerns the classification, structure and representation theory of finite-dimensional Lie algebras (§2.1) and Lie superalgebras (§2.2). The field $\mathbb{K}$ is assumed to be $\mathbb{C}$. The aim of the chapter is to expose a sufficiently detailed summary of the established theory upon which relevant generalisations can be built in later chapters. Importantly this chapter serves as a reference point for notational conventions and canonical choices for bases, root systems and inner products.

## 2.1 Lie Algebras

The primary references for the standard definitions and results presented in this section are the detailed texts of Carter [15] and Humphreys [41] as well as the concise review on the topic by Belinfante and Kolman [6]. The sections concerning the representation theory, especially as regards the use of Young tableaux, also draw upon the work of Wybourne [78][77].

### 2.1.1 Definitions

#### Lie Algebra

A *Lie algebra* is a vector space $L$ over a field $\mathbb{K}$ with a bilinear multiplication $[\ ,\ ] : L \times L \to L$ satisfying the following axioms for all $x, y, z \in L$

- **Antisymmetry** $\quad [x, y] = -[y, x]$
- **Jacobi Identity** $\quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

The multiplication operator $[\ ,\ ]$ is often called the Lie bracket, the commutator or simply the bracket. A Lie algebra of dimension $n$ may be defined by a set of basis generators

$x_i \in L$, $i = 1, .., n$ with the brackets given by

$$[x_i, x_j] = c_{ij}{}^k x_k.$$

The constants $c_{ij}{}^k$ are called the *structure constants* of the Lie algebra. In terms of the structure constants we may express the antisymmetry of the algebra as

$$c_{ij}{}^k = -c_{ji}{}^k$$

and the Jacobi identity as

$$c_{ij}{}^l c_{kl}{}^m + c_{jk}{}^l c_{il}{}^m + c_{ki}{}^l c_{jl}{}^m = 0.$$

A *subalgebra* $S$ of a Lie algebra $L$ is a linear subspace $S \subset L$ such that $[S, S] \subset S$. The Lie bracket is bilinear satisfying

$$[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[X, Y] \quad \text{and} \quad [X, aY + bZ] = a[X, Y] + b[X, Z], \alpha, \beta \in k$$

which, together with the additive properties of the underlying vector space, give the Lie algebra the structure of a nonassociative ring. We define an *ideal* $I$ of $L$ to be a subspace of $L$ satisfying $[L, I] \subset I$. Due to antisymmetry we have $[L, I] = [I, L]$.

A subalgebra is called *solvable* if $L^{(k)} = 0$ for some positive integer $k$ where

$$L^{(1)} := [L, L] \quad \text{and} \quad L^{(k)} := [L^{(k-1)}, L^{(k-1)}].$$

A subalgebra is called *nilpotent* if $L^k = 0$ for some positive integer $k$ where

$$L^1 := [L, L] \quad \text{and} \quad L^k := [L, L^{k-1}].$$

It follows that a nilpotent Lie subalgebra is always solvable however the converse is not true.

We now introduce two important definitions. A *semisimple Lie algebra* is a Lie algebra with no solvable ideals, other than zero. A *simple Lie algebra* is a non-abelian Lie algebra with no proper ideals at all. All simple Lie algebras, with the exception of the one-dimensional trivial Lie algebra, are semisimple. These definitions and their role in the decomposition and classification of Lie algebras will be discussed later in the chapter.

Any associative algebra $A$ can be made into a Lie algebra by defining the Lie bracket

$$[x, y] := xy - yx \qquad \text{for all } x, y \in A$$

where $xy$ denotes multiplication in $A$ of the elements $x$ and $y$. This definition of the commutator is clearly antisymmetric and the fulfillment of the Jacobi identity is easily verified. Let $M_n(\mathbb{K})$ be the associative algebra of all $n \times n$ matrices over the field $k$. We define $gl_n(\mathbb{K})$ to be the Lie algebra constructed out of $M_n(\mathbb{K})$ in this way.

### Representations and Modules

A *representation* of a Lie algebra $L$ is a linear map $\rho : L \to gl_n(\mathbb{K})$ satisfying $\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$ for all $x, y \in L$. Two representations are *equivalent* if they are related by a similarity transform.

An *L-module* is a vector space $V$ over $\mathbb{K}$ with a multiplication, or module action, given by

$$\begin{gathered} L \times V \to V \\ (x, v) \mapsto x.v, \end{gathered}$$

where $x.v$ is linear in $x$ and $v$ and $[x, y].v = x.(y.v) - y.(x.v)$. We note that such modules are technically left-modules, owing to the left action of the multiplication, however it will be assumed that all modules are left-modules unless otherwise stated. Every $L$-module gives a representation of $L$. Let $\{e_1, e_2, ..., e_n\}$ be a basis for the $L$-module $V$ then

$$x.e_j = \sum_{i=1}^{n} \rho_{ij}(x)e_i$$

and $\rho : x \mapsto (\rho_{ij}(x))$ is a representation of $L$. Different bases of the same $L$-module give equivalent representations. An $L$-submodule is a subspace $U$ of $V$ that is also a module; that is $L.U \subset U$. A module is called an *irreducible module* if it has no submodules other than itself and zero. The corresponding representation is also called *irreducible*.

An important example is the *adjoint module* where the Lie algebra $L$ acts as a module over itself. The module action is defined by the Lie bracket

$$ad_x : y \mapsto [x, y] \qquad x, y \in L.$$

We call the representation which maps $x \in L$ to $ad_x \in gl_n(\mathbb{C})$, where $n = dimL$, the *adjoint representation.* The adjoint representation gives us an opportunity to view the Jacobi identity from a different perspective. Demanding that $ad_x$ is a representation is equivalent to demanding that the Jacobi identity be satisfied, viz.

$$\begin{aligned} & ad_{[x,y]}(z) && = (ad_x ad_y - ad_y ad_x)(z) \\ \Rightarrow\ & [[x, y], z] && = [x, [y, z]] - [y, [x, z]] \\ \Rightarrow\ & [[x, y], z] && = [x, [y, z]] + [y, [z, x]]. \end{aligned}$$

Given a fixed basis for $L$, we may write

$$ad_{x_i}(x_j) = [x_i, x_j] = c_{ij}{}^k x_k$$

thus matrix elements of basis generators in the adjoint representation may be expressed in terms of the structure constants $(ad_{x_i})_j{}^k = c_{ij}{}^k$.

### The Killing Form

The *Killing form* is a bilinear form $\langle\ ,\ \rangle : L \times L \to \mathbb{C}$ defined by

$$\langle x, y\rangle = tr(ad_x ad_y).$$

In a fixed basis $x_i$ we may use the Killing form to define a metric, $g_{ij}$, which may be calculated in terms of the structure constants

$$g_{ij} = \langle x_i, x_j\rangle = c_{ik}{}^l c_{jl}{}^k. \tag{2.1}$$

A Lie algebra $L$ is semisimple if and only if the Killing form is non-degenerate. One useful function of the metric tensor is that it enables a dual basis for semi-simple Lie algebras. Let $g^{ij}$ be the inverse of $g_{ij}$ then the corresponding dual basis $x^i \in L^*$ is given by

$$x^i = g^{ij} x_j, \tag{2.2}$$

which satisfies $\langle x^i, x_j \rangle = \langle g^{ik} x_k, x_j \rangle = \delta^i{}_j$

### Cartan Subalgebras

To further investigate the structure and ultimately the classification of Lie algebras, we introduce a certain maximal abelian subalgebra called the *Cartan subalgebra*. Formally a Cartan subalgebra is any subalgebra $H$ equal to its own normaliser $N(H) = \{x \in L : [h, x] \in H, \forall h \in H\}$. The dimension of $H$ is called the *rank* $l$ of the Lie algebra. Every finite-dimensional Lie algebra has a Cartan subalgebra and if $H_1$ and $H_2$ are both Cartan subalgebras of $L$ then one may be obtained from the other through an inner automorphism of $L$ (that is, there exists $a \in L$ such that $H_1 = \{exp(ad_a)h,\, h \in H_2\}$.) Cartan subalgebras are nilpotent.

### Weights and Roots

Let $H$ be a nilpotent Lie algebra and let $V$ be a $H$-module. A linear form $\mu \in H^*$ is called a *weight* of $V$ if there exists a non-zero $v \in V$ such that

$$h.v = \mu(h)v \quad \forall h \in H. \tag{2.3}$$

We call $v$ a *weight vector*.

A *weight subspace* of $V$ is the set $V_\mu = \{v \in V : h.v = \mu(h)v\,, \forall h \in H\}$. When $V$ is a finite dimensional irreducible representation then it may be decomposed into a direct sum of its weight subspaces

$$V = \bigoplus_{\mu} V_\mu.$$

(We note that a more general result exists for decomposing arbitrary modules over nilpotent Lie algebras however this requires the definition of a generalised weight vector $v$ satisfying $(h - \mu(h))^n.v = 0$ for some integer $n$.)

We defer further discussion on the structure and classification of irreducible representations to section (2.1.3) and examine now the special case of the adjoint module. A semisimple Lie algebra $L$ may be viewed as a module over one of its Cartan subalgebras $H$; using the above decomposition we may write

$$L = \bigoplus_{\alpha} L_\alpha = H \oplus (\bigoplus_{\alpha \neq 0} L_\alpha),$$

where $L_\alpha = \{x \in L : [h, x] = \alpha(h)x\,, \forall h \in H\}$ and, in particular, $L_0 = H$. We call this the *Cartan decomposition* of $L$ and instead of weights we call the $\alpha \in H^*$ the *roots* of $L$ with respect to $H$. When $L$ is semisimple each subspace $L_\alpha$, $\alpha \neq 0$, is one-dimensional. It follows that the number of non-zero roots is equal to the dimension of $L$ minus the rank. It is the analysis of roots and their properties that is exploited to classify the simple Lie algebras. We state now a few useful properties for future reference.

Let $L$ be a semisimple Lie algebra of rank $l$ and $H$ be a Cartan subalgebra of $L$. Let $\Delta$ denote the set roots of $L$ with respect to $H$. Now $\Delta$ spans $H^*$ and there exists a subset of roots $\Pi = \{\alpha_i \,,\, i = 1, .., l\}$ called the *simple roots* which are a basis for $H^*$. An additional property of the simple roots is that they allow a decomposition of the root space into two subsets $\Delta^+$ and $\Delta^-$. We define the set of *positive roots* $\Delta^+ = \{\alpha \in \Delta : \alpha = n^i\alpha_i \,,\, n^i \in \mathbb{Z}^+\}$ and the set of *negative roots* $\Delta^- = \{\alpha \in \Delta : \alpha = n^i\alpha_i \,,\, n^i \in \mathbb{Z}^-\}$. Every non-zero root $\alpha$ is either a positive or a negative root such that $\Delta = \Delta^+ \cup \Delta^-$ and $\Delta^+ \cap \Delta^- = \{0\}$. Furthermore if $\alpha \in \Delta^+$ then $-\alpha \in \Delta^-$. Finally we note that in order to construct a set of simple of roots with the above properties the root system must first be expressed in terms of an ordered basis for $H^*$. The choice of ordering fixes the subsets $\Delta^\pm$ and $\Pi$.

**Bases**

The Cartan decomposition tells us that we may choose a basis for $L$ that consists of $l$ mutually commuting elements belonging to $H$ and one *root vector* $e_\alpha$ belonging to each of the root spaces $L_\alpha$. The restriction of the Killing form to $H$ gives the following isomorphism from $H^*$ to $H$

$$\alpha \mapsto h_\alpha \quad \text{defined by} \quad \alpha(x) = \langle h_\alpha, x\rangle \quad \forall x \in H.$$

This in turn induces an inner product on $H^*$ given by

$$\langle \alpha, \beta\rangle = \langle h_\alpha, h_\beta\rangle.$$

In order to write down a concrete set of commutation relations we must fix a scaling for the basis elements in $L$. One way to do this is to fix the normalisation of the roots. The canonical choice $\sum\limits_\alpha \langle\alpha,\alpha\rangle = l$ yields the *Cartan-Weyl basis*

$$\begin{array}{rcl} [h_{\alpha_i}, h_{\alpha_j}] &=& 0 \\ [h_{\alpha_i}, e_{\pm\alpha}] &=& \pm\alpha(h_{\alpha_i})e_{\pm\alpha} \\ [e_\alpha, e_{-\alpha}] &=& \langle\alpha,\alpha\rangle h_\alpha \\ [e_\alpha, e_\beta] &=& \left\{ \begin{array}{cl} N_{\alpha\beta}e_{\alpha+\beta} & \qquad \alpha+\beta \text{ is a root} \\ 0 & \qquad \alpha+\beta \text{ is not a root.} \end{array} \right. \end{array} \tag{2.4}$$

The $N_{\alpha\beta}$ are given by $N^2_{\alpha\beta} = \frac{1}{2}n(1+m)\langle\alpha,\alpha\rangle$ where in the chain,

$$\beta - m\alpha, \beta-(m-1)\alpha, ..., \beta+(n-1)\alpha, \beta+n\alpha,$$

the first and last terms are not roots.

We now introduce an alternative basis which allows the commutation relations to be expressed in an even simpler form than the above. We begin by defining the *Cartan matrix* $A = (A_{ij})$ where

$$A_{ij} := \frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_i,\alpha_i\rangle} \qquad i,j = 1, ..., l \tag{2.5}$$

It turns out that the simple Lie algebras are uniquely characterised by their Cartan matrix which encodes enough information to generate the full set of roots. By choosing an appropriate rescaling of the Cartan-Weyl generators, the defining relations of the Lie algebra are able to be expressed in terms of elements of the Cartan matrix. The *Chevalley* basis consists of the following Cartan elements and simple root vectors

$$\begin{array}{lcll} h_i & := & \frac{2h_{\alpha_i}}{\langle\alpha_i,\alpha_i\rangle} & \\ e_i & := & e_{\alpha_i} & \text{(simple raising element)} \\ f_i & := & \frac{2e_{-\alpha_i}}{\langle\alpha_i,\alpha_i\rangle\langle e_{\alpha_i},e_{-\alpha_i}\rangle} & \text{(simple lowering element.)} \end{array} \tag{2.6}$$

These generators satisfy the defining relations

$$\begin{array}{rcl} [h_i, h_j] & = & 0 \\ [h_i, e_j] & = & A_{ij}e_j \\ [h_i, f_j] & = & -A_{ij}f_j \\ [e_i, f_j] & = & \delta_{ij}h_i \\ [e_i, e_j] & =: & e_{ij}\ (i < j) \\ [f_i, f_j] & =: & f_{ij}\ (i < j) \end{array} \tag{2.7}$$

where the non-simple root vectors $e_{ij}$ and $f_{ij}$ are defined by the commutators of simple ones. The remaining commutators involving non-simple roots vectors are able to be determined employing the Jacobi identity and the following (Serre) relations [41]:

$$\begin{array}{ll} (ad_{e_i})^{1-A_{ij}}(e_j) = 0 & \quad i \neq j \\ (ad_{f_i})^{1-A_{ij}}(f_j) = 0 & \quad i \neq j \end{array}$$

In addition to simplifying the commutation relations, we shall see that the Chevalley basis facilitates some computational aspects of the representation theory of Lie algebras (see section 2.1.3.)

### 2.1.2 Summary of Classification

The Levi theorem states that any finite-dimensional Lie algebra $L$ may be decomposed as

$$L = S \oplus R$$

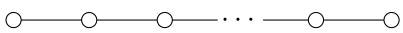
where $S$ is a semisimple Lie algebra and $R$ is a maximal solvable ideal of $L$ called the *radical.* Furthermore, a Lie algebra is semisimple if and only if it is isomorphic to a direct sum of non-trivial simple Lie algebras. Since $[L, R] \subset R$ it follows that $R$ is an $L$-module. In view of the Levi theorem, the classification of Lie algebras is reduced to the classification of simple and solvable Lie algebras. Classification of the latter is an extremely difficult problem; to date only partial results exist for low dimensions and those with certain structural properties [67]. In the following sections we restrict our attention to the former, providing a brief summary of the classification of complex simple Lie algebras and aspects of their representation theory.

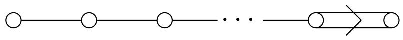
The classification of semisimple Lie algebras amounts to the classification of admissible structure constants which, in view of the Chevalley basis, is equivalent to classifying all possible Cartan matrices. Given the definition of the Cartan matrix, the problem can be restated as the search for possible simple root systems $\Pi$. The restriction of the bilinear form to $H^*$ can be further restricted to $H^*_{\mathbb{R}}$, the real restriction of the complex vector space $H^*$ spanned by $\Pi$. This yields a bilinear form that is positive definite and as such the simple roots are a basis for an $l$-dimensional Euclidean space. Thus, the characterisation of possible root spaces becomes a geometric problem with the inner product on $H^*_{\mathbb{R}}$ taking the form $\langle\alpha, \beta\rangle = |\alpha||\beta|cos\theta$.

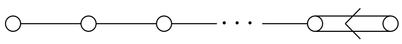
As a consequence of the fact that an arbitrary root may be expressed as an integer combination of simple roots it turns out that the possible angles $\theta$ between any two distinct simple roots is constrained to be one of $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$. Each of these angles corresponds to a fixed ratio of lengths between the two simple roots. These geometric restrictions in $H^*_{\mathbb{R}}$ translate into the following constraints on the Cartan matrix [15].

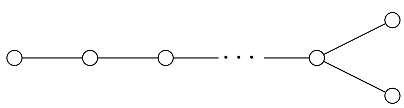
1. $A_{ii} = 2$ for all $i$.
2. $A_{ij} \in \{0, -1, -2, -3\}$if $i \neq j$.
3. If $A_{ij} = -2$ or $-3$ then $A_{ji} = -1$.
4. $A_{ij} = 0$ if and only if $A_{ji} = 0$.

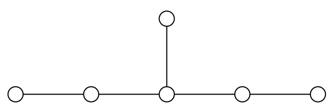
Finally, this information may encoded in a graphical form called the Dynkin diagram which is constructed as follows. For each simple root a vertex is drawn as a small circle. The $i^{th}$ and $j^{th}$ vertices are connected by $n_{ij} = A_{ij}A_{ji}$ edges where $n_{ij} \in \{0, 1, 2, 3\}$. The set of all possible Dynkin diagrams is shown below and these constitute a complete classification of the complex simple Lie algebras.

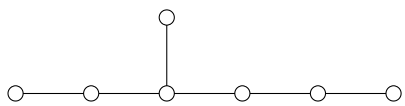
$A_\ell$ 

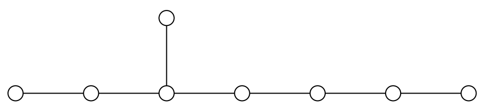
$B_\ell$ 

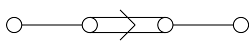
$C_\ell$ 

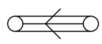
$D_\ell$ 

$E_6$ 

$E_7$ 

$E_8$ 

$F_4$ 

$G_2$ 

There are four general series $A_l, B_l, C_l$ and $D_l$ corresponding to the four classical families of Lie algebras $sl(l+1)$, $so(2l+1)$, $sp(2l)$ and $so(2l)$ respectively. In addition there are five exceptional algebras $E_6, E_7, E_8, F_4$ and $G_2$. The arrows point to the smaller of the two simple roots connected by two or more edges. It is possible to sketch disconnected Dynkin diagrams in cases where $n_{ij} = 0$. These correspond to semisimple Lie algebras with each connected part corresponding to a simple Lie algebra in the direct sum decomposition.

#### The Classical Root Systems

It is convenient to express the roots belonging to the $l$-dimensional root space of the classical Lie algebras in terms of an orthonormal basis $\varepsilon_i$, $i = 1, .., l+1$ over $\mathbb{R}$, with the inner product given by $\{\varepsilon_i, \varepsilon_j\} = \delta_{ij}$. There exists a constant $\kappa$ such that

$$\langle \alpha_i, \alpha_j \rangle = \kappa \{\alpha_i, \alpha_j\}. \tag{2.8}$$

In this basis the simple roots are given by

$$\begin{aligned} \alpha_i = &\quad \varepsilon_i - \varepsilon_j \qquad i, j = 1, .., l-1, \\ \alpha_\ell = &\begin{cases} \varepsilon_l - \varepsilon_{l+1} & \text{for type } A_l \\ \varepsilon_l & \text{for type } B_l \\ 2\varepsilon_l & \text{for type } C_l \\ \varepsilon_{l-1} + \varepsilon_l & \text{for type } D_l \end{cases} \end{aligned} \tag{2.9}$$

and the entire root system is given by

$$\Delta = \begin{cases} \pm(\varepsilon_i - \varepsilon_j) & 1 \le i < j \le l+1 & \text{for type } A_l \\ \pm\varepsilon_i \pm \varepsilon_j & 1 \le i < j < l & \text{for type } B_l \\ \pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon & 1 \le, i < j, k \le l & \text{for type } C_l \\ \pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon & 1 \le, i < j, k \le l & \text{for type } D_l. \end{cases} \tag{2.10}$$

As an example, we note that for the $A_l$ series every root $\alpha \in \Delta$ satisfies $\{\alpha, \alpha\} = 2$ and the Cartan matrix elements can be written as

$$A_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} = \frac{2\kappa\{\alpha_i, \alpha_j\}}{\kappa\{\alpha_i, \alpha_i\}} = \{\alpha_i, \alpha_j\} = \{\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j\}$$

without the need to determine the value of $\kappa$. In the general case, it is unnecessary to determine $\kappa$, thus the above list of simple roots expressed in $\varepsilon$-basis allows the commutation relations (2.7) to be easily computed.

### 2.1.3 Representation Theory

#### Highest Weight Representations

A important step in developing the representation of Lie algebras is to investigate the action of Lie algebra elements on tensor product modules. Let $L$ be a semisimple Lie algebra and let $V$ and $W$ be $L$-modules. Then for $x \in L$, $v \in V$ and $w \in W$ we have the following module action on $V \otimes W$

$$x.(v \otimes w) = (x.v) \otimes w + v \otimes (x.w).$$

Under this action $V \otimes W$ satisfies the definition of a module resulting from the existence of the coproduct homomorphism given by $x \mapsto x \otimes 1 + 1 \otimes x$. We note that definition is also consistent with the definition of the module action of the corresponding Lie group, that is $G \to G \otimes G$. Let us now restrict $V$ to an irreducible module and consider $L \otimes V$ as an $H$ module where $H$ is a Cartan subalgebra of $L$. Both $L$ and $V$ admit weight space decompositions, thus for $h \in H$, $e_\alpha \in L_\alpha$ and $v_\mu \in V_\mu$ we may write

$$\begin{aligned} h(e_\alpha \otimes v_\mu) &= (h.e_\alpha) \otimes v_\mu + e_\alpha \otimes (h.v_\mu) \\ &= (\alpha(h) + \mu(h))(e_\alpha \otimes v_\mu). \end{aligned}$$

It follows that $L_{\pm\alpha}.V_\mu \subset V_{\mu\pm\alpha}$ and we call $e_\alpha$ and $e_{-\alpha}$ *raising* and *lowering* operators respectively. We generalise this by taking both $V$ and $W$ to be arbitrary irreducible $H$-modules. It follows that $V_\mu \otimes V_\nu \subset (V \otimes W)_{\mu+\nu}$ and every weight in the tensor product $V \otimes W$ can be expressed as the sum of a weight of $V$ and a weight of $W$.

The simple root system $\Pi$ determines an ordered basis for $H^*$, which in turn orders the set of weights of an arbitrary module $V$. When $V$ is a finite-dimensional representation there must exist a *highest weight* which we denote $\lambda$. We call $V_\lambda$ the *maximal weight space* of $V$. It follows that $V_\lambda = \{v \in V : e_\alpha.v = 0,\ \forall \alpha \in \Delta^+\}$; each element $v \in V_\lambda$ is called a *highest weight vector*. When $L$ is a semisimple Lie algebra the maximal weight space is one-dimensional such that $V_\lambda = \mathbb{C}v_\lambda$.

For each Lie algebra of rank $l$ we define the *basic weights* $\omega_i$, $i = 1, ..., l$ satisfying

$$\frac{2\langle \omega_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}. \tag{2.11}$$

This condition is equivalent to $\omega_i(h_j) = \delta_{ij}$. The basic weights are a basis for $H^*$. Any weight $\mu$ of a finite dimensional module is an integer combination of the basic weights called an *integral weight*. A *dominant integral weight* is an integral weight $\lambda$ such that

$$\lambda = a_1\omega_1 + ... + a_l\omega_l, \quad a_i \in \mathbb{Z},\ a_i \geq 0.$$

The *theorem of the highest weight* states that a module $V = V(\lambda)$ is finite-dimensional if and only if the highest weight $\lambda$ is dominant integral. Furthermore, finite-dimensional irreducible representations are uniquely characterised, up to isomorphism, by their highest weight. The coefficients $a_i$ of the highest weight are called the *Dynkin indices* of a finite-dimensional irreducible representation.

Finally, we note that the transformation from the basis of simple roots $\alpha_i$ to the basis of basic weights $\omega_i$ is given by the Cartan matrix

$$\alpha_i = \sum_j A_{ji}\omega_i \tag{2.12}$$

$$\omega_i = \sum_j (A^{-1})_{ji}\alpha_i. \tag{2.13}$$

**Some Universal Constructions**

A prerequisite for further investigation of the representation theory of Lie algebras are the definitions of certain algebraic structures with so called universal properties. Let $X$ be a vector space over $\mathbb{K}$ of finite dimension $n$. We define $T^0 = \mathbb{C}$, $T^1 = X$, $T^2 = X \otimes X, \ldots, T^m = X \otimes \ldots \otimes X$ ($m$ copies). Next we define

$$T(X) = T^0 \oplus T^1 \oplus T^2 \oplus \cdots$$

and introduce an associative product $T^m \times T^n \to T^{m+n}$ defined via

$$(x_1 \otimes \cdots \otimes x_m).(y_1 \otimes \cdots \otimes y_n) = x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n$$

for $x_i, y_j \in X$ and extended linearly to all of $T(X)$. $T(X)$ is an associative algebra called the *tensor algebra* of $X$. $T(X)$ is *universal* in the following sense. Let $\sigma$ denote the inclusion map $\sigma : X \to T(X)$ such that $\sigma(x_i) = x_i \in T^1$ and let $A$ be any associative algebra over $\mathbb{C}$. Given any linear map $\theta : X \to A$ there exists a unique homomorphism of associative algebras $\phi : T(X) \to A$ such that $\theta = \phi \circ \sigma$.

We now introduce some factor algebras of $T(X)$. Let $I_S$ be the ideal of $T(X)$ generated by elements of form $x \otimes y - y \otimes x$ for all $x, y \in X \cong T^1$ and let $I_E$ be the ideal of $T(X)$ generated by elements of form $x \otimes y + y \otimes x$ for all $x, y \in X \cong T^1$. We define

$$S(X) = T(X)/I_S \qquad \text{The Symmetric Algebra} \tag{2.14}$$

$$E(X) = T(X)/I_E \qquad \text{The Exterior Algebra.} \tag{2.15}$$

In $S(X)$ we write $x \vee y$ to denote the coset $x \otimes y + I_S$. $S(X)$ inherits the graded structure of $T(X)$ where each graded subspace $S^k$ has dimension equal to the binomial coefficient $\binom{k+n-1}{k}$ and a basis given by the ordered products $x_{i_1} \vee x_{i_2} \vee \cdots \vee x_{i_k}$, $i_1 \leq i_2 \leq i_3 \leq \ldots$. Similarly for $E(X)$ we write $x \wedge y$ to denote the coset $x \otimes y + I_E$. In this case, the ideal contains elements of the form $\mathbb{C}x \otimes x$. Consequently $E(X)$ is finite-dimensional with $dimE^k = \binom{n}{k}$ $k \leq n$ and $dimE^k = 0$, $k \geq n$. A basis for $E(X)$ is the set of strictly ordered products $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k}$, $i_1 < i_2 < i_3 < \ldots$ where $k \leq n$.

As far as the representation theory of Lie algebras is concerned the most important factor algebra of the tensor algebra is the *universal enveloping algebra* $U(L)$. Let $X = L$ and let $I$ be the ideal of $T(L)$ generated by elements of form $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in L$. We set

$$U(L) = T(L)/I.$$

In this case the universal property inherited from $T(L)$ ensures that for every Lie algebra representation $\rho : L \to gl_n(k)$ there exists a unique homomorphism of associative algebras from $U(L)$ into $gl_n(k)$. In other words the associative algebra $U(L)$ shares the same representation theory as the Lie algebra $L$.

**Theorem 2.1.1 (Poincaré-Birkhoff-Witt (PBW) theorem for Lie algebras)** *Let $x_i$, $i = 1, .., n$ be a basis for $L$. The set of ordered monomials*

$$(x_1)^{i_1} \cdots (x_n)^{i_n}, \; i_k \in \{0, 1, 2, 3, \ldots\}$$

*are a basis for $U(L)$.*

**The Structure of Finite Dimensional Representations**

Let $L$ be a semisimple Lie algebra and $H$ a Cartan subalgebra. $\Pi$ denotes a fixed simple root system and the $\Delta^{\pm}$ the positive and negative roots spaces with respect to $\Pi$. The Cartan decomposition can be used to write the Lie algebra as

$$L = H \oplus L_- \oplus L_+$$

where $L_\pm = \{\mathbb{C}e_\alpha : \alpha \in \Delta^\pm\}$. We have seen that a finite-dimensional irreducible module is characterised by its highest weight $\lambda$. We have also seen that raising and lowering operators act on module elements by shifting their weight. When $V$ is an irreducible module it must be the case that every weight vector $v_\mu \in V$ is able to be obtained, up to a factor, from any other weight vector by repeated application of the root vectors. If this were not the case then a non-trivial submodule would exist and $V$ would not be irreducible. The repeated module action of root vectors introduces a tensor multiplication of Lie algebra elements. For example

$$e_\alpha(e_\beta.v_\mu) = e_\alpha e_\beta.v_\mu = kv_{\mu+\alpha+\beta}$$

may be viewed as a mapping $(L \otimes L) \times V \to V$ where $k \in \mathbb{C}$. The definition of the module action requires that $[x, y].v = x.(y.v) - y.(x.v) = (xy - yx).v$ which implies that $[x, y] \in L$ and $x \otimes y - y \otimes x \in L \otimes L$ are equivalent under the module action. The coset structure of the universal enveloping algebra $U(L)$ makes this equivalence explicit and it follows that $V(\lambda) = U(L)v$ for any $v \in V$. For each of the subalgebras $L_\pm$ and $H$ we may define corresponding subalgebras $U(L_\pm)$ and $U(H)$ of $U(L)$. The PBW theorem implies $U(L) = U(L_-)U(H)U(L_+)$ and we may write $V(\lambda) = U(L).v_\lambda = U(L_-).v_\lambda$.

There are $l$ basic $L$-modules $V(\omega_i) = U(L_-)(\omega_i)$, $i = 1, .., l$ whose highest weights are the basic weights. The structure of the basic modules are easier to determine than those of general modules as the termination of the sequence

$$v_{\omega_j}, e_{\alpha_i}v_{\omega_j}, (e_{\alpha_i})^2 v_{\omega_j}, ..., (e_{\alpha_i})^q v_{\omega_j} = 0 \tag{2.16}$$

called the $\alpha_i$-ladder through $\omega_j$, where $(e_{\alpha_i})^n v_\lambda$ for $n < q$ are all non-zero, is given by $q = -A_{ij}$. It turns out that the Cartan matrix contains enough information to determine all $\alpha$-ladders through any weight belonging to a basic module and the structure of the entire module can be computed. The structure of an arbitrary module $V(\lambda) = V(\sum a_i\omega_i)$ can then be constructed using the *Cartan composition*

$$V(\lambda) = U^-.(\underbrace{v_{\omega_1} \otimes ... \otimes v_{\omega_1}}_{a_1 times} \otimes ...... \underbrace{\otimes v_{\omega_l} \otimes ... \otimes v_{\omega_l}}_{a_l times}) \tag{2.17}$$

where the coefficients $a_i$ are the Dynkin indices.

The final structural problem to be resolved is that of tensor product modules. Let $V(\lambda)$ and $V(\mu)$ be finite dimensional irreducible modules of $L$ then the tensor product module has a unique decomposition

$$V(\lambda) \otimes V(\mu) = \bigoplus_\nu m^\nu_{\lambda\mu} V(\nu) \tag{2.18}$$

where $V(\nu)$ are finite dimensional-irreducible $L$-modules. This equation is called the *Clebsch-Gordan Series* and the coefficients $m^{\nu}_{\lambda\mu}$ are the multiplicities. The determination of these coefficients is an important problem in representation theory. This topic will be discussed in the following section in the context of $gl(n)$ modules.

### 2.1.4 Representation Theory of $gl(n)$

The classical Lie algebras are all subalgebras of the non-semisimple general linear Lie algebra $gl(n)$. The defining representation for $gl(n)$ is given by the $n \times n$ elementary matrices $e^i{}_j$ which have a 1 in the $(i, j)$-entry and 0 elsewhere. We denote by $E^i{}_j$, $i, j = 1, .., n$ the Gel'fand generators which are the set of abstract operators satisfying the same commutation relations as the elementary matrices namely

$$[E^i{}_j, E^k{}_l] = \delta^k{}_j E^i{}_l - \delta^i{}_l E^k{}_j. \tag{2.19}$$

We may express the generators of the classical Lie algebras in this basis. For example the Cartan elements and the simple root vectors of the Chevalley basis (2.7) for the $A_l$ series may be written as

$$\begin{aligned} h_i =& E^i{}_i - E^{i+1}{}_{i+1} \\ e_i =& E^i{}_{i+1} \\ f_i =& E^{i+1}{}_i. \end{aligned} \tag{2.20}$$

Similar relations exist for the other series of classical Lie algebras. One reason to study the representation theory of $gl(n)$ is that it is easier to perform certain computations, such as the determination of Clebsch-Gordon series coefficients and dimension formula, than is the case for the classical Lie algebras directly. One method for doing this is via the construction of Young diagrams.

A finite-dimensional irreducible representation of $gl(n)$ is characterised by a highest weight vector $\lambda \in H^*$. The $n$-dimensional Cartesian basis $\epsilon_i$, $i = 1, .., n$ is a basis for $H^*$ and each $\lambda$ may be uniquely written as $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ where $\lambda_i \leq \lambda_{i+1}$. For each such $\lambda$ a diagram may be drawn consisting of $n$ rows of left-aligned boxes with each row having $\lambda_i$ boxes. Together these boxes form a shape and each box may then be filled with an integer between 1 and $\sum \lambda_i$ to form a so-called semi-standard tableau (this requires that the integers to be non-decreasing along each row and strictly increasing down each column.) Corresponding to each shape are a finite set of semi-standard tableaux and each tableau corresponds to a unique monomial expression of the form $x_1^{k_1} x_2^{k_2} \cdots$, comprising $\sum \lambda_i$ indeterminants $x_j$ for which the index $j$ appears $k_j$ times in the corresponding tableau. The sum of all these monomials for a given shape $\lambda$ form the symmetric function $s_\lambda$, called a *Schur function.* The set of Schur functions, where $n$ ranges over all of $\mathbb{Z}$ and $\lambda$ ranges over all allowed shapes, are a basis for the ring of symmetric functions $\Lambda$, and indeed each $s_\lambda$ itself is an irreducible character, when the indeterminates $x_j$ are identified as class functions of $gl(n)$. The product of Schur functions in $\Lambda$ takes the form

$$s_\lambda s_\mu = \sum_\nu m^{\nu}_{\lambda\mu} s_\nu$$

where $m^{\nu}_{\lambda\mu}$, in this context called the *Littlewood-Richardson coefficients*, are the multiplicities of the corresponding Clebsch-Gordon Series (compare (2.18)). This coincidence arises as the Schur functions themselves turn out to be the characters (see §2.1.5) of the irreducible representations $V(\lambda)$ of $gl(n)$. The relationship between representations of $gl(n)$ and the Schur functions as a basis for $\Lambda$ translates the difficult problem of computing the multiplicities of the Clebsch-Gordon Series into a combinatorial problem which can be solved diagrammatically for low dimensional cases or, more importantly, by implementing appropriate algorithms on a computer (see for example [78]).

Having established a method for performing these calculations for $gl(n)$ certain modification rules can be imposed to solve the corresponding representation theoretic problems of the classical subalgebras. Families of distinct representations of $gl(n)$ may correspond to a single representation of a classical subalgebra. In the case of $A_l$ the Dynkin indices of the corresponding representation $A_l-$module are given by the differences in row lengths of the associated Young diagram, $a_i = \lambda_i - \lambda_{i+1}$, $\lambda_i, i = 1, .., n-1$ (cf. equation 2.20.) The modification rule for $A_l$ involves simply deleting Young tableaux with $n$ or more rows. Similar modification rules exist for the other classical subalgebras.

### 2.1.5 Casimir Operators and Characters

The set $Z = \{z \in U(L) : zu = uz\ \forall u \in U(L)\}$ is called the *centre* of $U(L)$. $Z$ is a $U(L)$-subalgebra isomorphic to the polynomial ring over $\mathbb{C}$ in $l$ variables where $l$ is the rank of $L$. It is easily shown that elements of $Z$ act on a finite-dimensional irreducible $U(L)$-module $V(\lambda)$ by scalar multiplication. For any $z \in Z$ this scalar value $\chi_\lambda(z) \in \mathbb{C}$ is fixed by the highest weight $\lambda$ of the module.

The function $\chi_\lambda : Z \to \mathbb{C}$ is a one-dimensional representation of the $Z$ called the *central character* of $V(\lambda)$. A closely related concept is the (infinitesimal) character of a module $M$ which maps every Cartan element $h \in H$ to the scalar value $\chi_M(h) = Tr_M e^h$. The character uniquely identifies finite-dimensional irreducible modules up to isomorphism and contains enough information to determine their dimension as well as the direct sum decomposition of tensor product modules. This was illustrated in the previous subsection via the connection between characters and Schur functions. The well-known Harish-Chandra isomorphism identifies elements in $H$ with elements in $Z$.

An important element of $Z$ is the *universal (or quadratic) Casimir*

$$c = x^i x_i \tag{2.21}$$

where $x_1, ..., x_k$ are any basis for a semi-simple Lie algebra $L$ and $x^i = g^{ij} x_j$ are the dual elements with respect the Killing form. $c$ is basis independent and acts on finite-dimensional irreducible modules with the fixed eigenvalue

$$\begin{aligned}\chi_\lambda(c) &= \langle \lambda + \rho, \lambda + \rho \rangle - \langle \rho, \rho \rangle \\ &= \langle \lambda, \lambda + 2\rho \rangle \end{aligned} \tag{2.22}$$

where

$$\rho = \omega_1 + \cdots + \omega_l = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha. \tag{2.23}$$

In the case of $gl(n)$ the set of $n$ linearly independent Casimir invariants $\mathsf{C}_k$, $k = 1, ..., n$ can be succinctly obtained using the formula

$$\mathsf{C}_k = Tr[(e^i{}_j \otimes E^i{}_j)^k] \tag{2.24}$$

where $e^i{}_j \otimes E^i{}_j$ is the $n \times n$ matrix with $E^i{}_j$ at each $(i, j)$-entry and $k$ denotes the matrix power.

### 2.1.6 Tensor Representations

We have seen the representation theory of Lie algebras reduced from general modules to irreducible modules and then from irreducible modules to basic modules. In this final section we reduce the construction of basic modules down to a single module called the *fundamental module*. As has been the case in previous sections, the technique will be exemplified for $gl(n)$ as well as the $A_l$ series, however, it should be understood that the approach can be adapted to the other classical algebras.

The fundamental module of $A_l$ is the basic module $V = V(\omega_1)$. All of the basic modules $V(\omega_k)$ $k = 1, ..., l$ can be obtained by projecting out the antisymmetric subspace of $\otimes^k V$. This is simply the $k^{th}$-exterior power $V(\omega_k) = \wedge^k V$.

The introduction of the $k^{th}$ tensor power of the $L$-module $V$ introduces a natural action of the permutation group $S_k$ on the same module. Irreducible representations of the permutation group $S_k$ are in one-to-one Young diagrams; with basis enumerated by a corresponding set of Young tableau consisting of $k$ boxes where the row lengths represent cycle lengths, from largest to smallest, and the $k$ integers which fill the boxes simply correspond to numbers within each cycle. Two elements of $S_k$ are conjugate if and only if they have the same cycle types, thus the conjugacy classes are characterised by Young diagrams consisting of $k$ boxes. For example, the group element $(123)(4)(5) \in S_5$, which cyclically permutes the indices 123 while fixing the other two, corresponds to the Young shape $(3, 1, 1)$.

It is a well known result that the number of irreducible representations of a finite group is equal to the number of its conjugacy classes, therefore corresponding to each Young diagram is a unique irreducible $S_k$-representation $S_k(\lambda)$.

Returning to $gl(n)$ we have seen that each irreducible representation may also be uniquely mapped to a Young diagram. The correspondence of $gl(n)$ representations with those of $S_k$ is formally known as *Schur-Weyl duality*; this is a special case of the double commutant theorem. The theorem states that the commutative action of the two algebras on an arbitrary $k$-fold tensor product module of $gl(n)$ decomposes the module into a pairwise direct sum of irreducible representations given by

$$\bigotimes^k V = \bigoplus_\lambda S_k(\lambda) \otimes V(\lambda)$$

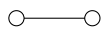
where the sum is indexed by $\lambda$ which runs over all allowable Young diagrams consisting of $k$ boxes and at most $n$ rows. In other words, representations of $S_k$ and $gl(n)$ may be put into one-to-one correspondence where each may be used to characterise the other and the dimension of one determines the multiplicity of the other.

Schur-Weyl duality allows the idea of constructing the basic irreducible modules from the fundamental $V$ to be extended to that of obtaining any irreducible module by identifying the appropriately symmetrised subspace of $\otimes^k V$. The process of Cartan composition can be bypassed altogether by using the permutation group to define the *Young projectors* $P_\lambda$ which satisfy $V(\lambda) = P_\lambda \otimes^k V$. The projection operator corresponding to a given Young tableau is given by $P_\lambda = MN$ where $M = \sum m$ is the sum of all the group elements $m$ that permute integers in the same row and $N = \sum sgn(n)n$ is the signed sum of all group elements $q$ that permute integers in the same row.

Finally we note that the Young projectors give a complete resolution of the identity. In other words that the set of Young diagrams composed of $k$ boxes characterises the complete decomposition of $\otimes^k V$ into its irreducible components.

### 2.1.7 Examples

$\mathbf{L = A_2 \cong sl(3)}$



Starting with (2.9) and (2.10), the root system is determined by

$$\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3\} \quad \text{and} \quad \Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} = -\Delta^-. \tag{2.25}$$

Using (2.5), the Cartan matrix and its inverse are

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad \text{and} \qquad A^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

The generators of $L$ are obtained using (2.4) and (2.6),

$$\textbf{Cartan-Weyl basis: } h_{\alpha_1}, h_{\alpha_2}, e_{\pm\alpha_1}, e_{\pm\alpha_2}, e_{\pm(\alpha_1+\alpha_2)} \tag{2.26}$$

$$\textbf{Chevalley basis: } h_1, h_2, e_1, f_1, e_2, f_2, e_{12}, f_{12}. \tag{2.27}$$

Using (2.7), the non-zero commutation relations in the Chevalley basis are

$$\begin{array}{llll} [h_1, e_1] = 2e_1 & [h_1, e_2] = -e_2 & [h_2, e_1] = -e_1 & e_{12} = [e_1, e_2] \\ [h_2, e_2] = 2e_2 & [e_1, f_1] = h_1 & [e_2, f_2] = h_2 & f_{12} = [f_1, f_2] \end{array} \tag{2.28}$$

where the remaining commutators involving $e_{12}$ and $f_{12}$ are easily evaluated using the Jacobi identity and the Serre relations.

There are two basic weights (2.11),

$$\begin{array}{lll} \omega_1 & = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2 & = \varepsilon_1 - \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \\ \omega_2 & = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2 & = -\varepsilon_3 + \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3). \end{array} \tag{2.29}$$

The corresponding basic modules (2.17) are $V(\omega_i) = U(L_-).v_{\omega_i}$ . The weight space of each is computed by considering $\alpha$-ladders (2.16) commencing at the highest weight. The weight space of the three-dimensional fundamental module $V = V(\omega_1)$ is

$$\begin{aligned}
\lambda_1 &= \omega_1 && = \varepsilon_1 - \frac{1}{3}\sum \varepsilon_k \\
\lambda_2 &= \omega_1 - \alpha_1 && = \varepsilon_2 - \frac{1}{3}\sum \varepsilon_k \\
\lambda_3 &= \omega_1 - \alpha_1 - \alpha_2 && = \varepsilon_3 - \frac{1}{3}\sum \varepsilon_k .
\end{aligned} \tag{2.30}$$

Comparing (2.29) and (2.30), we see that the lowest weight of the $V$ is the negative of the highest of $V(\omega_2)$. We identify $V^* = V(\omega_2)$ as the three dimensional *contragredient* (or *dual*) module of $V$. The weight space of $V^*$ is the negative of the weight space of $V$.

Tensor product decompositions can be handled explicitly in terms of a fixed basis for each of the modules involved, see below, or alternatively using (rank independent) tensorial methods that exploit the symbolic manipulation of Young diagrams, see the next example, $sl(n)$.

Let us examine the decompositions of $V \otimes V$ and $V \otimes V^*$. We fix the following bases

$$V = \langle v^1, v^2, v^3 \rangle_{\mathbb{K}} \tag{2.31}$$
$$V^* = \langle v_1, v_2, v_3 \rangle \tag{2.32}$$

where each basis element $v^i$ and $v_i$ has weight $\lambda_i$ and $-\lambda_i$ respectively and $\langle\ \rangle_{\mathbb{K}}$ denotes the linear span over $\mathbb{K}$. Expressing the generators of $L$ in terms of Gel'fand generators (2.20) allows the module action on $V$ and $V^*$ to be succinctly written as

$$E^i{}_j . v^k = \delta^k{}_j v^i \quad \text{and} \quad E^i{}_j . v_k = \delta^i{}_k v^j .$$

This action is consistent with (2.20),(2.3) and (2.30).

$V \otimes V$ decomposes into its symmetric (+) and antisymmetric (−) subspaces given by

$$\begin{aligned}
(V \otimes V)_+ &= \langle v^i \otimes v^j + v^j \otimes v^i \rangle_{\mathbb{K}} \qquad (dim = 6) \\
(V \otimes V)_- &= \langle v^i \otimes v^j - v^j \otimes v^i \rangle_{\mathbb{K}} \qquad (dim = 3).
\end{aligned}$$

where each is an irreducible $L$-module. $(V \otimes V)_+$ has highest weight $2\omega_1$ and $(V \otimes V)_-$ has highest weight $-\omega_2$ and is isomorphic to $V^*$.

$V \otimes V^*$ also decomposes into two irreducible $L$-modules. The first is the eight-dimensional traceless part spanned by the elements

$$\langle v^i \otimes v_j - \delta^i{}_j \sum v^i \otimes v_i \rangle_{\mathbb{K}} .$$

This is the adjoint module with highest weight $\omega_1 + \omega_2$. The second is the one-dimensional trace $\langle v^i \otimes v_i \rangle$ which is the identity representation. This example serves to make explicit the relationship between the indices of Gel'fand generators and their associated weight. Each generator $E^i{}_j$ is associated with the vector $v^i \otimes v_j \in V \otimes V^*$ with weight $\lambda_i - \lambda_j = \varepsilon_i - \varepsilon_j$.

To complete the example, we examine the quadratic Casimir and its eigenvalues. The first step is to determine the components of the metric tensor (2.1). Using the structure constants in the Chevalley basis (2.28), ordered as in (2.27), we obtain

$$(g_{ij}) = 6\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad (g^{ij}) = \frac{1}{6}\begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.33}$$

Using the above, together with the relations (2.28), the quadratic Casimir (2.21) is

$$c = \frac{1}{6}\left[\frac{2}{3}\left((h_1)^2 + h_1h_2 + (h_2)^2\right) + e_1f_1 + f_1e_1 + e_2f_2 + f_2e_2 + e_{12}f_{12} + f_{12}e_{12}\right]. \tag{2.34}$$

The fixed eigenvalue of $c$ acting on a finite-dimensional irreducible module with highest weight $\lambda$ is given by the character (2.22)

$$\chi_\lambda(c) = \langle\lambda, \lambda + 2\rho\rangle = \frac{1}{6}\{\lambda, \lambda + 2\rho\} \tag{2.35}$$

where $\langle,\rangle$ denotes the Killing form and $\{,\}$ is the inner product on the Euclidean space spanned by $\varepsilon_1, .., \varepsilon_n$. Comparing (2.35) and (2.8) we identify $\kappa = \frac{1}{6}$ which could be determined by comparing, for example, $g_{11} = Tr(ad(h_1)ad(h_1))$ and $\{\alpha_1, \alpha_1\}$.

Let us compute some examples values of $\chi_\lambda(c)$. We start with

$$\rho = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_{12}) = \varepsilon_1 - \varepsilon_3$$

and compute the following:

$$\begin{aligned}
\{\omega_1, \omega_1\} =& \{\omega_2, \omega_2\} && = \frac{2}{3} \\
\{\omega_1, \omega_2\} =& \{\omega_2, \omega_1\} && = \frac{1}{3} \\
\{\omega_1, 2\rho\} =& \{\frac{2}{3}\varepsilon_1 - \frac{1}{3}\varepsilon_2 - \frac{1}{3}\varepsilon_3, 2\varepsilon_1 - 2\varepsilon_3\} && = \frac{6}{3} \\
\{\omega_2, 2\rho\} =& \{\frac{1}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_2 - \frac{2}{3}\varepsilon_3, 2\varepsilon_1 - 2\varepsilon_3\} && = \frac{6}{3}.
\end{aligned}$$

We use the above, together with (2.35), to evaluate the following characters:

$$\begin{aligned}
\chi_V(c) &= \frac{1}{6}\{\omega_1, \omega_1 + 2\rho\} = \frac{4}{9} \\
\chi_{ad}(c) &= \frac{1}{6}\{\omega_1 + \omega_2, \omega_1 + \omega_2 + 2\rho\} = 1 \\
\chi_{a_1\omega_1 + a_2\omega_2}(c) &= \frac{1}{9}\{a_1\omega_1 + a_2\omega_2, a_1\omega_1 + a_2\omega_2 + 2\rho\} \\
&= \frac{1}{9}\left((a_1)^2 + a_1a_2 + (a_2)^2 + 3a_1 + 3a_2\right).
\end{aligned}$$

$\mathbf{L = A_{n-1} = sl(n)}$

The root system is determined by (2.9) and (2.10), giving

$$\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \,|\, i = 1, .., n-1\} \quad \text{and} \quad \Delta^+ = \{\varepsilon_i - \varepsilon_j \,|\, i < j \in 1, .., n-1\} = -\Delta^-.$$

The Cartan matrix and its inverse (2.5) are

$$A_{n-1} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots & -1 \\ & & & & -1 & 2 \end{pmatrix} \qquad A^{n-1} = \frac{1}{n} \begin{pmatrix} n-1 & n-2 & \cdots & 2 & 1 \\ & & \cdots & & \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \tag{2.36}$$

We will use as a basis for $L$ the Gel'fand generators (2.20), satisfying the commutation relations given by (2.19). The basic weights, computed from the inverse Cartan matrix, are [15]

$$\omega_i = \frac{1}{n}\left((n-1)(\varepsilon_1 + \cdots + \varepsilon_i) - i(\varepsilon_{i+1} + \cdots + \varepsilon_n)\right) \tag{2.37}$$

The fundamental representation $V$ and its contragredient $V^*$ have a basis (generalising (2.31))

$$V = \langle v^i \rangle_{\mathbb{K}} \qquad i = 1, ..., n \tag{2.38}$$
$$V^* = \langle v_i \rangle_{\mathbb{K}} \qquad i = 1, ..., n \tag{2.39}$$

where the upper (lower) index $i$ corresponds to the weight $\lambda_i = (-)\varepsilon_i - \frac{1}{n}\sum_k \varepsilon_k$.

We begin the analysis of tensor product representations decompositions by identifying

$$V = \mathbf{n} = (1) = \Box \qquad\qquad V^* = \overline{\mathbf{n}} = (1^n) = \begin{array}{c} \Box \\ \vdots \\ \Box \end{array} \quad (n-1)\, boxes$$

where the notation $\mathbf{k}$ indicates a representation of dimension $k$. When the dimension does not uniquely characterise the representation then extra notations are used; for example, $\overline{\mathbf{k}}$ denotes contragredient representations. The tensor product decomposition of $V \otimes V$ may be written as

$$\Box \otimes \Box = \begin{array}{|c|}\hline \phantom{x} \\ \hline \phantom{x} \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline \phantom{x} & \phantom{x} \\ \hline \end{array} \tag{2.40}$$

where the right-hand side is constructed by adding together the boxes on the left-hand side into all allowable combinations subject to certain modification rules. Rather than detail the rules for allowable diagrams the reader is directed to the reference [77] (this theory is applied in Section 5.3 for the calculation of zero-step modules). Diagrams consisting of a single vertical column correspond to antisymmetric representations (recall these are the fundamental representations) and diagrams consisting of a single horizontal row correspond to symmetric representations.

To complete the example, we again examine the quadratic Casimir and its eigenvalues. The metric (2.33), computed in the previous example, generalises to

$$
(g_{ij}) = 2n \left( \begin{array}{c|ccc}
(A_{ij}) & & & \\
\hline
 & \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} & & \\
\cline{2-4}
 & & \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} & \\
\cline{2-4}
 & & & \ddots
\end{array} \right) \tag{2.41}
$$

where $A_{ij}$ is the Cartan matrix. The Casimir elements in this basis becomes

$$
C = \frac{1}{2n} \left[ A^{ij} H_i H_j + \sum_{i \neq j} E^i_j E^j_i \right] \tag{2.42}
$$

where $A^{ij}$ is the inverse Cartan matrix. The eigenvalues of $c$ are given by (2.22).

## 2.2 Lie Superalgebras

The first part of this section gives an overview of well-known results and is based upon the original publications due to Kac [48] [49], in addition to the succinct summary [28] and also [29]. Additional references are cited in text.

### 2.2.1 Definitions

A *superalgebra* is a $\mathbb{Z}_2$-graded algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ satisfying

$$
A_{\bar{0}} A_{\bar{0}} \subset A_{\bar{0}}, \quad A_{\bar{0}} A_{\bar{1}} \subset A_{\bar{1}}, \quad A_{\bar{1}} A_{\bar{1}} \subset A_{\bar{0}}.
$$

Elements of $A_{\bar{0}}$ and $A_{\bar{1}}$ are called even (degree 0) and odd (degree 1) respectively. $A$ is called commutative if $xy = (-1)^{|x||y|} yx$ for all $x, y \in A$, where $|x|$ is the degree of the element $x \in L$.

A *Lie superalgebra* over $\mathbb{K}$ is a superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ with a product $[\ ,\ ]$ satisfying the following axioms:

- **$\mathbb{Z}_2$-gradation**

$$
[L_i, L_j] \subset L_{i+j} \quad (i, j \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\})
$$

- **Graded anticommutativity**

$$
[x_i, x_j] = -(-1)^{|x_i||x_j|} [x_j, x_i]
$$

- **Jacobi Identity**

$$(-1)^{|x_i||x_k|}[x_i,[x_j,x_k]]+(-1)^{|x_j||x_i|}[x_j,[x_k,x_i]]+(-1)^{|x_k||x_j|}[x_k,[x_i,x_j]]=0 \quad (2.43)$$

An alternative expression of these conditions takes the following form:

- The even part $L_{\bar{0}}$ is an ordinary Lie algebra.
- The odd part $L_{\bar{1}}$ is an $L_{\bar{0}}$-module.
- The multiplication of pairs of odd elements determines a homomorphism of $L_{\bar{0}}$-modules $\psi : S^2 L_{\bar{1}} \to L_{\bar{0}}$ satisfying the condition $\psi(a,b)c+\psi(b,c)a+\psi(c,a)b=0$.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space and denote by $End\,V$ the algebra of endomorphisms of $V$. $End\,V$ naturally acquires the structure of an associative superalgebra where $End_i\,V = \{\phi \in EndV|\, \phi(V_i) \subset V_{i+1}\}$. In analogy with the Lie algebra $gl(n)$ we define the Lie superalgebra $gl(m|n)$ as the superalgebra $End\,V$ endowed with the product

$$[x,y] = xy - (-)^{|x||y|}yx$$

where $dim\,V_{\bar{0}} = m$ and $dim\,V_{\bar{1}} = n$.

A *representation* of a Lie superalgebra $L$ is a homomorphism $\pi : L \to gl(m|n)$. In other words $\pi$ preserves the $\mathbb{Z}_2$-grading and satisfies

$$\pi([x,y]) = [\pi(x),\pi(y)].$$

The underlying vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ requires the structure an $L$-module under the action $x.v = \pi(x)v$, satisfying $[x,y].v = x.(y.v) - (-1)^{|x||y|}y.(x.v)$. In the homogeneous basis $e_1, ..., e_m, e_{m+1}, ..., e_{m+n}$ of the $L$-module $V$, where $dim\,V_{\bar{0}} = m$ and $dim\,V_{\bar{1}} = n$, an element $x \in L$ is represented by the matrix

$$\pi(x) = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \in gl(m|n) \quad (2.44)$$

where $a, b, c$ and $d$ are $m \times m$, $m \times n$, $n \times m$ and $n \times n$ matrices respectively. Even elements have a block diagonal form ($b = c = 0$) and odd elements are composed of off-diagonal blocks ($a = d = 0$).

The *supertrace* $str(x) = tr(a) - tr(d)$ is defined for all $x \in gl(m|n)$. For each representation $\pi$ of $L$ one can define the bilinear form $B_\pi(x,y) = str(\pi(x)\pi(y))$. When $\pi$ is the adjoint representation $ad : L \times L \to L$, defined by $ad_x(y) = [x,y]$, then $\langle x,y\rangle = B_{ad}(ad_x, ad_y)$ is called the *Killing form*.

The definitions *nilpotency* and *solubility* are identical to those of Lie algebras. A Lie superalgera is solvable if and only if $L_{\bar{0}}$ is solvable. A *subalgebra* $K = K_{\bar{0}} \oplus K_{\bar{1}}$ is a vector subspace of $L$ that is closed with respect to the Lie product and satisfies $K_i \subset L_i$. An *ideal* $I$ is a subalgebra of $L$ satisfying $[L,I] \subset I$.

A Lie superalgebra is called *simple* if it does not contain any non-trivial ideals. A Lie superalgebra $L$ is called *semisimple* if it does contain any non-trivial solvable ideals, however unlike the Lie algebra case this does not imply that $L$ can be written as a direct sum of simple Lie superalgebras.

A $\mathbb{Z}$-*grading* of a Lie superalgebra $L$ is a decomposition of $L$ into a direct sum of finite-dimensional $\mathbb{Z}_2$-graded subspaces $L_i$, such that

$$L = \bigoplus_{i\in\mathbb{Z}} L_i, \qquad \text{where } [L_i, L_j] \subset L_{i+j}.$$

A $\mathbb{Z}$-grading is said to be *consistent* if $L_{\bar{0}} = \bigoplus L_{2i}$ and $L_{\bar{1}} = \bigoplus L_{2i+1}$.

### 2.2.2 Classification of Simple Lie Superalgebras

Given that the even part of a Lie superalgebra is an ordinary Lie algebra, the classification of Lie Superalgebras amounts to the determination of a suitable $L_{\bar{0}}$-module $L_{\bar{1}}$ for which a homomorphism $\psi(\{L_{\bar{1}}, L_{\bar{1}}\}) \to L_{\bar{0}}$, consistent with the Jacobi identity, is defined. In a manner analogous to the classification of Lie algebras, a semisimple Lie superalgebra may be obtained from an arbitrary Lie superalgebra via factorisation by a unique maximal solvable ideal. In turn, a semisimple Lie superalgebra permits a description, although not a direct sum decomposition, in terms of finite-dimensional simple Lie superalgebras. In 1977, Kac [48] completed the classification of finite-dimensional simple Lie superalgebras which is briefly described below.

A *classical* Lie superalgebra is a simple Lie superalgebra in which the $L_{\bar{0}}$-module $L_{\bar{1}}$ is completely reducible. It is said to be of type I if $L_{\bar{1}} = L_{-1} \oplus L_1$ where $L_{\pm 1}$ are each irreducible. In this case $[L_{-1}, L_1] = L_{\bar{0}}$, $[L_{\pm 1}, L_{\pm 1}] = 0$ and $L = L_{-1} \oplus L_0 \oplus L_1$ is a consistent $\mathbb{Z}$-grading of $L$ where $L_{\bar{0}} = L_0$. The superalgebra is said to be of type II if $L_{\bar{1}}$ is an irreducible $L_{\bar{0}}$-module.

A classical Lie superalgebra $L$ is called *basic* if there exists a non-degenerate invariant bilinear form on $L$ (invariance of a bilinear form $B$ implies $B([x, y], z) = B(x, [y, z])$ for all $x, y, z \in L$). There are four infinite families of basic Lie superalgebras: $A(m, n)$, $B(m, n)$, $C(n)$ and $D(m, n)$ where the even part of each of these has some relationship to the corresponding series of simple Lie algebras (see table below). Additionally there are three exceptional basic Lie superalgebras $F(4)$, $G(3)$ and $D(2, 1; \alpha)$. Lie superalgebras which are classical, but not basic, are called *strange* and there are two infinite families of these denoted by $P(n)$ and $Q(n)$. Table 2.1 lists the classical Lie superalgebras and their $\mathbb{Z}_2$-graded subspaces.

The remaining simple Lie superalgebras are those which are not classical. These are termed the *Cartan type* Lie superalgebras which are classified into four infinite families denoted by $W(n)$ with $n \geq 2$, $S(n)$ with $n \geq 3$, $\tilde{S}(n)$ and $H(n)$ with $n \geq 4$. In this thesis, we will be primarily concerned with type I basic classical Lie superalgebras.

| **Superalgebra** $L$ | $L_{\bar{0}}$ | $L_{\bar{1}}$ |
|---|---|---|
| $A(m-1,n-1)$ | $A_{m-1} \oplus A_{n-1} \oplus U(1)$ | $(m,\bar{n}) \oplus (\bar{m},n)$ |
| $A(n-1,n-1)$ | $A_{n-1} \oplus A_{n-1}$ | $(n,\bar{n}) \oplus (\bar{n},n)$ |
| $C(n+1)$ | $C_n \oplus U(1)$ | $(2n) \oplus (2n)$ |
| $B(m,n)$ | $B_m \oplus C_n$ | $(2m+1,2n)$ |
| $D(m,n)$ | $D_m \oplus C_n$ | $(2m,2n)$ |
| $F(4)$ | $A_1 \oplus B_3$ | $(2,8)$ |
| $G(3)$ | $A_1 \oplus G_2$ | $(2,7)$ |
| $D(2,1;\alpha)$ | $A_1 \oplus A_1 \oplus A_1$ | $(2,2,2)$ |

Table 2.1: $\mathbb{Z}_2$-*gradation of the basic classical Lie superalgebras* [28]

### 2.2.3 Structure of Basic Classical Lie Superalgebras

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a basic classical Lie superalgebra. A *Cartan subalgebra* $H$ is simply a Cartan subalgebra of $L_{\bar{0}}$ and the *rank* of $L$ is $dim\, H$. $L$ admits a root space decomposition

$$L = \bigoplus_{\alpha \in H^*} L_\alpha \quad \text{where} \quad L_\alpha = \{x \in L \,|\, [h,x] = \alpha(h),\, h \in H\}$$

The root system is $\Delta = \{\alpha \in H^* | L_\alpha \neq 0\}$. The set of even roots $\Delta_{\bar{0}} = \{\alpha \in H \,|\, L_\alpha \cap L_{\bar{0}} \neq \emptyset\}$ is the root system of the even part. The set of odd roots $\Delta_{\bar{1}} = \{\alpha \in H \,|\, L_\alpha \cap L_{\bar{1}} \neq \emptyset\}$ is the weight system of $L_{\bar{1}}$ as an $L_{\bar{0}}$-module. The root space $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$ spans $H^*$. An inner product on $H^*$ can be defined in precisely the same way as for Lie algebras by first restricting to $H$, the bilinear form on $L$.

One of the main differences between the characterisation of the root systems of Lie superalgebras, compared to that of Lie algebras, is that there are in general, many different ineq uivalent simple roots systems associated with each Lie superalgebra $L$. Each simple root system $\Pi$ can be uniquely identified with a Borel decomposition $L = N^- \oplus H \oplus N^+$, where $N^\pm$ are subalgebras such that $[H, N^\pm] \subset N^\pm$ and $dim\, N^+ = dim\, N^-$.

Amongst the set of inequivalent simple root systems, there exists a unique one called the *distinguished* simple root system which contains just one odd root $\alpha_s$. If $L = \bigoplus_{i \in \mathbb{Z}} L_i$ is the distinguished $\mathbb{Z}$-gradation then the even simple roots are the simple root system of $L_0$ and the simple odd root $\alpha_s$ is the lowest weight of $L_1$ as an $L_0$-module. The distinguished root systems of the basic Lie superalgebras are given in table 2.2.

For each simple root system $\Pi$, one can define the Cartan matrix $A_{ij} = (\alpha_i, \alpha_j)$; this allows the commutation relations to be expressed in a canonical form much like the Chevalley basis for Lie algebras. The Cartan matrix may also be encoded into Dynkin diagrams according to a set of rules similar, albeit slightly more complicated, to those used for Lie algebras (see [48]).

| **Superalgebra $L$** | **Distinguished simple root system $\Pi$** |
|---|---|
| $A(m-1,n-1)$ | $\varepsilon_1-\varepsilon_2,...,\varepsilon_{m-1}-\varepsilon_m,\varepsilon_m-\delta_1,\delta_1-\delta_2,...,\delta_{n-1}-\delta_n$ |
| $B(m,n)$ | $\delta_1-\delta_2,...,\delta_{n-1}-\delta_n,\delta_n-\varepsilon_1,\varepsilon_1-\varepsilon_2,...,\varepsilon_{m-1}-\varepsilon_m,\varepsilon_m$ |
| $B(0,n)$ | $\delta_1-\delta_2,...,\delta_{n-1}-\delta_n,\delta_n$ |
| $C(n)$ | $\varepsilon_1-\delta_1,\delta_1-\delta_2,...,\delta_{n-1}-\delta_n,2\delta_n$ |
| $D(m,n)$ | $\varepsilon_1-\varepsilon_2,...,\varepsilon_{m-1}-\varepsilon_m,\varepsilon_m-\delta_1,\delta_1-\delta_2,...,\delta_{n-1}-\delta_n$ |
| $F(4)$ | $\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta),-\varepsilon_1,\varepsilon_1-\varepsilon_2,\varepsilon_2-\varepsilon_3$ |
| $G(3)$ | $\delta+\varepsilon_1,\varepsilon_2,\varepsilon_3-\varepsilon_2$ |
| $D(2,1;\alpha)$ | $\varepsilon_1-\varepsilon_2-\varepsilon_3,2\varepsilon_2,2\varepsilon_3$ |

Table 2.2: *Distinguished simple root systems of the basic classical Lie algebras. The vectors $\varepsilon_1,...,\varepsilon_m$ and $\delta_1,...,\delta_n$ are mutually orthogonal and satisfy $\varepsilon_i.\varepsilon_i=1$ and $\delta_j.\delta_j=-1$. Roots of the form $\varepsilon_i-\delta_j$ have zero length and correspond only to odd roots* [48].

### 2.2.4 Representation Theory

Finite-dimensional irreducible representations of the basic classical Lie superalgebras are classified by highest weight $\lambda \in H^*$, in much the same way as Lie algebras. Indeed, the general approach due to Kac [49] employs the universal enveloping algebra of $L$ to induce these representations from finite-dimensional irreducible representations of the even subalgebra.

The existence of a single odd root $\alpha_s$ in the distinguished simple root system complicates the general description of the structure of induced representations. For certain values of the Dynkin index $a_s := (\lambda, \alpha_s) \in \mathbb{C}$, the induced module is no longer irreducible and must be factorised by a certain maximal submodule. These representations are called *atypical* in contrast to *typical* representations which do not suffer this pathology.

In the atypical case, the determination of the maximal submodule and the subsequent structure of the irreducible factor module (Kac module) is a difficult problem (see [42]). A modified construction due to Gould [37] circumvents this by constructing both typical and atypical modules directly from a certain $L_{\bar{0}}$-module, contained within every submodule of the induced module.

In this section, we review the construction of Kac modules followed by Gould's modified construction for both typical and atypical modules. Finally, we provide a detailed illustration of the theory which is applied to the example $sl(2|1)$.

#### PBW Theorem

Let $T(L)$ be the tensor algebra generated by $L = L_{\bar{0}} \oplus L_{\bar{1}}$. $T(L)$ acquires the structure of a superalgebra where the $\mathbb{Z}_2$-graded degree of the homogenous element $w_1 \otimes ... \otimes w_k \in T^k$ is given by $\sum_{i=1}^{k} |w_i|$ for $w_i \in L$, an arbitrary basis element. Let $I$ be the ideal in $T(L)$ generated by elements of the form

$$[x, y] - x \otimes y + (-1)^{|x||y|} y \otimes x.$$

We define the *universal enveloping algebra* $U(L) = T(L)/I$. The PBW theorem for Lie algebras is easily generalised to Lie superalgebras.

**Theorem 2.2.1 (PBW basis for Lie superalgebras)** *Let $x_1, .., x_m$ be a basis for $L_{\bar{0}}$ and let $y_1, ..., y_n$ be a basis for $L_{\bar{1}}$. The elements of the form*

$$(x_1)^{k_1} \cdots (x_m)^{k_m} (y_1)^{l_1} \cdots y_n^{l_n}, \text{ where } k_i \in \{0, 1, 2, ...\} \text{ and } l_i \in \{0, 1\}$$

*form a basis for $U(L)$.*

For brevity we have dropped the explicit reference to the tensor product $\otimes$. A generalisation of the PBW theorem, which applies to a much broader class of non-commutative algebras, is given in Section 4.4.

One consequence of the PBW theorem is that it allows the enveloping algebra to be expressed in various factorised forms. Using a given direct sum decomposition of $L$ such as the $\mathbb{Z}_2$-gradation or Borel decomposition for example, one may write

$$U(L) = U(L_{\bar{0}})U(L_{\bar{1}}) \quad \text{or} \quad U(L) = U(N^-)U(L_0)U(N^+).$$

### Induced Representations of the Basic Lie Superalgebras of Type I

Let $L$ be a type I basic classical Lie superalgebra. $L$ admits a consistent $\mathbb{Z}$-gradation $L = L_- \oplus L_0 \oplus L_+$ where $L_{\bar{0}} = L_0$ and $L_{\bar{1}} = L_- \oplus L_+$ (note the abbreviated notation: $L_+ = L_{+1}$ and $L_- = L_{-1}$.) We fix a Borel decomposition $L = N^- \oplus H \oplus N^+$ such that associated simple root system is distinguished. Let $V_0(\lambda)$ be a finite-dimensional irreducible representation of $L_0$ characterised by the dominant integral weight $\lambda$. We turn $V_0(\lambda)$ into a $U(L_0 \oplus L_+)$-module by setting $L_+.V_0(\lambda) = 0$ and define

$$\overline{V} = Ind^{L}_{L_0 \oplus L_+} V_0(\lambda) = U(L) \otimes_{U(L_0 \oplus L_+)} V_0(\lambda) \tag{2.45}$$

where $\otimes_{U(L_0 \oplus L_+)}$ denotes factorisation of the tensor product space by the linear span of elements of the form $gh \otimes v - g \otimes hv$, $g \in U(L)$, $h \in U(L_0 \oplus L_+)$. This space acquires the structure of an $L$-module under the action

$$g(u \otimes v) = gu \otimes v, \; u \in U(L), \, v \in V_0(\lambda).$$

We make use of the factorisation $U(L) = U(L_-)U(L_0)U(L_+)$, to obtain

$$\begin{aligned} \overline{V}(\lambda) = U(L_-).V_0(\lambda) &= U(L_-)U(L_0).v_\lambda \\ &= \bigoplus_{l_i \in \{0,1\}} U(L_0)(y_1)^{l_1} \cdots (y_k)^{l_k}.v_\lambda \\ &= \bigoplus_{l_i \in \{0,1\}} V_0(\lambda - \sum_{i=1}^{k} l_i \alpha_i) \end{aligned} \tag{2.46}$$

where $\alpha_i \in \Delta_1^+$ is the root associated with each (negative) odd root vector $y_i \in L_-$, $k = dim\, \Delta_1^+$ and $v_\lambda$ is the unique highest weight vector of $V_0(\lambda)$. In other words, the induced module decomposes as the direct sum of irreducible $L_0$-modules, where the action of even elements generates states within each $L_0$-module and the odd elements shift between them.

#### Irreducible Modules and Atypicality Conditions

We begin with two canonical elements

$$T_+ = \prod_{y_i \in L_1} y_i \qquad T_- = \prod_{y_i \in L_{-1}} y_i \qquad \text{(enveloping algebra product)} \tag{2.47}$$

which are the (unique up to a sign) longest monomials possible in each of $U(L_{\pm 1})$ respectively. Consequently $T_\pm$ are one-dimensional $L_{\bar{0}}$-modules with highest weights $\pm 2\rho_1$ respectively, where

$$\rho_1 = \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha$$

is the usual half sum of positive odd roots.

For certain $\lambda \in H^*$ it may occur that $T_+T_-.V_0(\lambda) = 0$. In this case there exist weight vectors other than the highest weight vector $v_\lambda \in V_0(\lambda)$ that are annihilated by all positive root vectors. These weight vectors generate non-trivial $L$-submodules of $\overline{V}(\lambda)$. The approach taken by Kac to obtain the irreducible module with highest weight $\lambda$, is to factorise $\overline{V}(\lambda)$ by a unique maximal submodule $K(\lambda)$. The (irreducible) *Kac module* with highest weight $\lambda$ is defined to be

$$V(\lambda) = \overline{V}(\lambda)/K(\lambda)$$

where one simply has $V(\lambda) = \overline{V}(\lambda)$ in the case of typical modules. Every finite-dimensional irreducible representation of $L$ is of this form, where $\lambda$ is a dominant integral weight. The induced module $\overline{V}(\lambda)$ is typical (irreducible) when

$$T_+T_-.v = c \prod_{\alpha \in \Delta_1^+} (\lambda + \rho, \alpha).v \neq 0, \quad v \in V_0(\lambda). \tag{2.48}$$

In other words, $\overline{V}(\lambda)$ is atypical if one of $(\lambda + \rho, \alpha)$ is zero for $\alpha \in \Delta_1$. The *atypicality conditions* may be expressed succinctly in terms of the Dynkin indices. For example the atypicality conditions for the representation of $A(m, n)$ with highest weight $\lambda = (a_1, ..., a_{m+n-1})$ are

$$\sum_{k=1}^{n-1} a_k - \sum_{k+n+1}^{j} a_k + a_n = i + j - 2n$$

where $1 \leq i \leq n \leq j \leq m + n - 1$ [28].

#### A Modified Construction for Atypical Representations

The following modified construction due to Gould [37] allows a direct construction of the Kac modules in both typical and atypical cases, without the need to determine the maximal invariant submodule $K(\lambda)$. We begin by noting that one could equally well have constructed an induced $L$-module from a given (finite-dimensional) $L_0$-module $V_0(\lambda)$ by demanding that $L_-.V_0(\lambda) = 0$ (instead of $L_+.V_0(\lambda) = 0$) which leads to the module

$$\overline{V}_{(-)} = Ind_{L_0 \oplus L_-}^{L} V_0(\lambda) = U(L_+).V_0(\lambda) \tag{2.49}$$

which is cyclically generated by the lowest-weight vector of weight $\lambda_{(-)}$ where $\lambda_{(-)}$ is the lowest weight of $V_0(\lambda)$. It can be shown that every $L$-submodule of $\overline{V}_{(-)}(\lambda)$ (*resp.* $\overline{V}(\lambda)$) contains the $L_0$-module $T_+ \otimes V_0(\lambda)$ (*resp.* $T_- \otimes V_0(\lambda)$). From this it follows that the $L$-module generated from $T_+ \otimes V_0(\lambda)$ is necessarily irreducible with highest weight $\lambda + 2\rho_1$.

To construct an irreducible module with highest (dominant integral) weight $\lambda$, one simply needs to shift the weight of original $L_0$-module by $-2\rho_1$; it follows that every submodule of the corresponding (lowest weight) induced module

$$\overline{V}_{(-)}(\lambda - 2\rho_1) = U(L_+).V_0(\lambda - 2\rho_1)$$

contains the $L_0$-module $T_+.V_0(\lambda - 2\rho_1)$. Since $\langle \rho_1, \alpha \rangle = 0$ for $\alpha \in \Delta_{\bar{0}}^+$, $\lambda - 2\rho_1$ remains dominant integral and one obtains the finite-dimensional (highest weight) irreducible module by setting

$$V(\lambda) = U(L)T_+.V_0(\lambda - 2\rho_1). \tag{2.50}$$

The structure of the module can be determined directly from the commutation relations of $L$. In particular the decomposition of $V(\lambda)$ into a direct sum of irreducible $L_0$-modules can be resolved by considering expressions of the form

$$(y_1)^{l_1} \cdots (y_k)^{l_k} T_+.v \qquad l_i \in \{0, 1\}, v \in V_0(\lambda - 2\rho_1). \tag{2.51}$$

where the representation is of course typical if $T_- T_+.v \neq 0$. The task of identifying which $L_0$-modules decouple remains a difficult problem in the general case however for certain classes, such as $gl(n|1)$ and $sl(n|1)$, the theory of tensor projection operators and characteristic identities may be applied to significantly simplify the analysis [37][38]. The theory, originally due to Green [39] (see also [60][36]), allows equations (2.51) to be expressed in a factorised form such that the identification of the $L_0$-modules which are decoupled relates directly to those odd roots which satisfy $\langle \lambda + \rho, \alpha_i \rangle = 0$, $i = 1, .., n$, see (2.48). This method is exposed in detail in chapter 5 and to a lesser degree in the example that follows.

### 2.2.5 Example

**$\mathbf{A(1,0) \cong sl(2|1)}$**

The Lie superalgebra $sl(2|1) \cong sl(1|2)$ is a subalgebra of $gl(2|1)$ which we define below. Accordingly, we can write down a basis for $sl(2|1)$ in terms of the $gl(2|1)$-generators which greatly facilitates computations.

The generators of $gl(2|1)$ can be placed into a block-matrix array, where the diagonal blocks comprise the even subspace, and the off-diagonal blocks the odd subspace. The generators may be identified in the following way with the standard $gl(3)$ Gel'fand generators $E^i{}_j$, $i, j = 1, 2, 3$ (see 2.19):

$$\left(\begin{array}{cc|c} E^1{}_1 & E^1{}_2 & E^1{}_3 \\ E^2{}_1 & E^2{}_2 & E^2{}_3 \\ \hline E^3_1 & E^3{}_2 & E^3{}_3 \end{array}\right) = \left(\begin{array}{cc|c} E^1{}_1 & E^1{}_2 & \overline{Q}^1 \\ E^2{}_1 & E^2{}_2 & \overline{Q}^2 \\ \hline Q_1 & Q_2 & E^3{}_3 \end{array}\right) \tag{2.52}$$

The graded commutations relations of the generators on the right-hand side are

$$\begin{array}{llll} [E^i{}_j, E^k{}_l] & = \delta^k{}_j E^i{}_l - \delta^i{}_l E^k{}_j & [E^i{}_j, \overline{Q}^k] & = \delta^k{}_l \overline{Q}^i \\ [E^i{}_j, Q_k] & = -\delta^i{}_l Q_j & \{\overline{Q}^i, Q_j\} & = E^i{}_j + \delta^i{}_j E^3{}_3 \\ \{\overline{Q}^i, \overline{Q}^j\} & = \{Q_i, Q_j\} = 0 & & \end{array} \tag{2.53}$$

where the notation { , } is used instead of [ , ] to explicitly denote anticommutation in the case of $L_{\bar{1}} \times L_{\bar{1}}$ multiplication.

The Lie superalgebra $L = sl(2|1)$ comprises the subalgebra of $gl(2|1)$ matrices whose supertrace is zero. The following basis for $L = L_0 \oplus L_{-1} \oplus L_{+1}$ may be used

$$\begin{array}{ll} L_0 & = \{H, H_s, E^1{}_2, E^2{}_1\} \\ L_+ & = \{\overline{Q}^1, \overline{Q}^2\} \\ L_- & = \{Q_1, Q_2\} \end{array} \tag{2.54}$$

where $H = E^1{}_1 - E^2{}_2$ and $H_s = E^2{}_2 + E^3{}_3$. The root space is determined by equations (2.2) giving

$$\begin{array}{llll} \Delta_{\bar{0}}^+ & = \{\alpha = \varepsilon_1 - \varepsilon_2\} & \Delta_{\bar{1}}^+ = \{\bar{\alpha}_1 = \varepsilon_1 - \delta, \bar{\alpha}_2 = \varepsilon_2 - \delta\} \\ \Pi & = \{\alpha, \bar{\alpha}_1\} & \rho = -\varepsilon_2 + \delta = -\overline{\alpha}_2 \end{array} \tag{2.55}$$

It is customary in the physics literature to label weight vectors of finite-dimensional irreducible $L_0$-modules in terms of the two labels $|j, b\rangle$, representing spin and $gl(1)$-charge respectively. These satisfy

$$\frac{1}{2} H|j, b\rangle = j|j, b\rangle \tag{2.56}$$

$$(\frac{1}{2} H + H_s)|j, b\rangle = b|j, b\rangle,$$

where finiteness is guaranteed by requiring $j \in \frac{1}{2}\mathbb{Z}$. The weight $\mu$ associated with the weight vector $|\mu\rangle = |j, b\rangle$ may be written as

$$\begin{aligned} \mu &= j(\varepsilon_1 - \varepsilon_2) + b(-\varepsilon_1 - \varepsilon_2 + 2\delta) \\ &= (-b + j)\varepsilon_1 + (-b - j)\varepsilon_2 + 2b\delta. \end{aligned} \tag{2.57}$$

It follows that the odd generators may each be associated with following weights (and weight vectors):

$$\overline{Q}^1 = |\frac{1}{2}, -\frac{1}{2}\rangle \leftrightarrow \overline{\alpha}_1 \qquad \overline{Q}^2 = |-\frac{1}{2}, -\frac{1}{2}\rangle \leftrightarrow \overline{\alpha}_2 \tag{2.58}$$

$$Q_1 = |\frac{1}{2}, \frac{1}{2}\rangle \leftrightarrow -\overline{\alpha}_2 \qquad Q_2 = |-\frac{1}{2}, \frac{1}{2}\rangle \leftrightarrow -\overline{\alpha}_1 \tag{2.59}$$

where $\overline{Q}^1$ and $Q_2$ are the highest weight vectors of $L_+$ and $L_-$ respectively.

Our aim is to describe the structure of the irreducible $L$-modules $V(j, b)$ which are Kac modules satisfying the condition that all weight vectors $|j', b'\rangle$ belonging to the module satisfy

$j' \leq j$, $b' \leq b$. To achieve this, we need to take into account the fact that, although $\overline{\alpha}_1$ and $\overline{\alpha}_2$ are both positive roots in the distinguished system, the latter has both negative $j$ and $b$ values, see (2.58.) Consequently $\lambda - \overline{\alpha}_2$ raises both labels by $\frac{1}{2}$ and we must select the shifted $L_0$-module $V_0(j-\frac{1}{2}, b-\frac{1}{2})$ from which to induce the $L$-module $\overline{V}(j,b) = \overline{V}(\lambda')$ where $\lambda' = \lambda + \overline{\alpha}_2$. Using (2.46), we have [34][29]

$$\begin{aligned} \overline{V}(j,b) &= Ind^L_{L_0+L_+} V_0(j-\tfrac{1}{2}, b-\tfrac{1}{2}) \\ &= V_0(j-\tfrac{1}{2}, b-\tfrac{1}{2}) \oplus V_0(j-1,b) \oplus V_0(j,b) \oplus V_0(j-\tfrac{1}{2}, b). \end{aligned} \tag{2.60}$$

In the typical case the structure of the irreducible modules is simply $V(j,b) = \overline{V}(j,b)$ as given above. In this low rank example, the atypicality conditions may be easily determined from the commutation relations by explicitly evaluating

$$T^+T^-|j-\frac{1}{2}, b-\frac{1}{2}>= \overline{Q}^1\overline{Q}^2 Q_1 Q_2 |j-\frac{1}{2}, b-\frac{1}{2}>= 0.$$

It is simpler however to use the general rule (2.48) which leads to the two conditions

$$\begin{aligned} 0 &=< \lambda' + \rho, \overline{\alpha}_1 > =< (-b+j)\varepsilon_1 + (-b-j)\varepsilon_2 + 2\delta, \varepsilon_1 - \delta >= b+j \\ 0 &=< \lambda' + \rho, \overline{\alpha}_2 > =< (-b+j)\varepsilon_1 + (-b-j)\varepsilon_2 + 2\delta, \varepsilon_2 - \delta >= b-j \end{aligned}$$

where $\lambda' + \rho = \lambda + \overline{\alpha}_2 + \rho = \lambda$. In other words the Kac module $V(\lambda)$ is atypical when $b = \pm j$.

To complete the example we will determine the structure of the atypical modules. We reference two possible methods. The first, exemplified by Hunri and Morel [42], involves a somewhat tedious case-by-case examination of expressions of the form $\overline{Q}^i T^- |j-\frac{1}{2}, b-\frac{1}{2}>$ to identify candidate $L_0$-modules which may be decoupled from $\overline{V}(\lambda)$. A candidate module is then decoupled provided that $\overline{Q}^i T^- |j'b'\rangle = 0$, where $|j'b'\rangle$ are weight vectors other than the highest weight belonging to the candidate modules.

The second method, due to Gould [37], begins with the modified construction of section 2.2.4. Associated with each odd root $\alpha_r \in \Delta_1^+$ is a projection operator $P[r]$ which is expressed in terms of invariants and generators of $L_0$ (see section 5.2 for a detailed exposition of the theory). The projectors allow each odd generator to be resolved into the sum $Q_i = \sum_r Q[r]_i$, where $Q[r]_i = Q_j P[r]^j_i$, such that the action becomes

$$Q[r]_i T^+ V_0(\lambda - 2\rho_1) \in V_0(\lambda - \alpha_r). \tag{2.61}$$

Following Gould, the expressions (2.61) are evaluated with the aid of the characteristic identities of $L_0$. The result, see (5.21), is that irreducible modules $V(\lambda)$ contain the $L_0$-module $V_0(\lambda - \alpha_r)$ if and only if $\lambda - \alpha_r$ is dominant integral and $\langle \lambda + \rho, \alpha_r \rangle \neq 0$, $r = 1, 2$. Finally, since it is clear that $V_0(\lambda - \overline{\alpha}_1 - \overline{\alpha}_2) = T^-T^+.V_0(\lambda - 2\rho_1)$ belongs only to typical modules, we have

$$\begin{aligned} V(j,j) &= V_0(j,j) \oplus V_0(j-\tfrac{1}{2}, j+\tfrac{1}{2}) \\ V(j,-j) &= V_0(j,-j) \oplus V_0(j-\tfrac{1}{2}, -j-\tfrac{1}{2}). \end{aligned}$$

where one must select $V_0(j,j)$, rather than the shifted $L_0$-module $V_0(j-\frac{1}{2}, j-\frac{1}{2})$ from which to induce $V(j,j)$ else $V_0(j,j)$ would be decoupled [34] [29].

# Chapter 3

# REAL FORMS AND CHARACTERSTIC IDENTITIES

This chapter is composed of two sections and completes the mathematical background for the thesis.

The first section reviews the characterisation of the real forms of ordinary Lie algebras as well as some properties of their (unitary) representations. We restrict our attention to the $A_n$ series with a view to applying the theory in the final chapter which concerns quadratic deformations of the superconformal algebra $su(2,2|1)$.

The second section provides a review of the theory of characteristic identities of Lie algebras. As we have seen in the previous chapter, these play a key role in the modified construction of atypical modules for Lie superalgebras (see §2.2.4 and §2.2.5). Most importantly, in the context of this thesis, these identities make an appearance in the defining relations of the quadratic Lie superalgebras. Their presence gives rise to the remarkable existence of zero-step atypical representations; a property which is exploited in the superconformal application of the final chapter.

## 3.1 Real Forms

The classification of Lie Algebras is, in the first instance, reduced to the study of simple Lie algebras over the field of complex numbers. The complex field ensures algebraic closure which is required, in the general case, to determine the roots of the secular equation which governs the classification of admissible structure constants [32].

The process of complexification relates real Lie algebras to complex ones. In general a single complex Lie algebra generates several inequivalent real forms. In this section we examine the real forms $sl(n,\mathbb{R})$ and $su(p,q)$, $p+q=n$, related to the $A_{n-1}$ series of complex Lie algebras.

### 3.1.1 From Complex to Real Lie Algebras

Let $x_j, j = 1, .., n$ be a basis for a complex Lie algebra $L$. The real restriction, $L^R$ of the Lie algebra has dimension $2n$ and is the real Lie algebra spanned by the $2n$ linearly independent vectors elements $x_j$ and $\mathsf{i}x_j, j = 1, .., n$ where $\mathsf{i} = \sqrt{-1}$. If we can write a basis of $L^R$ in the form

$$L^R = L_0 \oplus \mathsf{i}L_0$$

then we call $L_0$ a *real form* of $L$. Every real Lie algebra can be obtained in this way [6].

A generator $X$ of a real form is said to be *compact* if and only if $\langle X, X\rangle = Tr(Ad(X)Ad(X)) < 0$, where $\langle\ ,\ \rangle$ is the Killing form on $L$ [33]. Of all the real forms of a given Lie algebra, there is a unique one called the *compact real form* of which all the generators are compact. A canonical choice for the generators of the compact real form may be given in terms of the Chevalley basis

$$\mathsf{i}h_j, i(e_\alpha + f_\alpha), e_\alpha - f_\alpha, \tag{3.1}$$

where $\alpha \in \Delta^+$. An arbitrary element of the compact real form belongs to the real span of this set.

The reason for identifying the compact real form as special is that it may be used to generate all of the real forms associated with a given complex Lie algebra.

**Example: The compact real form of $A_1 \cong sl(2,\mathbb{C})$**

Let us start with the Chevalley basis (2.6) for $sl(2,\mathbb{C})$ satisfying the relations

$$[h, e] = 2e \quad [h, f] = -2f \quad [e, f] = h. \tag{3.2}$$

We obtain the compact real form using the general prescription (3.1)

$$x'_1 = \mathsf{i}h \quad x'_2 = \mathsf{i}(e + f) \quad x'_3 = e - f.$$

One may easily verify the compactness of these generators by expressing each in terms of the Gelfand generators in the fundamental representation, see (2.20). The generators satisfy $[x'_i, x'_j] = -2\epsilon_{ijk}x'_k$, where $\epsilon_{ijk}$ is the totally antisymmetric tensor also known as the *Levi-Civita* symbol. In a physics context, these are easily recognised as a basis for $so(3,\mathbb{R})$ and we note the isomorphism of real forms $su(2) \cong so(3,\mathbb{R})$. More familiar relations are obtained by making the transformations

$$x_1 = \frac{1}{2}x'_1, x_2 = \frac{1}{2}x'_2 \text{ and } x_3 = -\frac{1}{2}x'_3 \tag{3.3}$$

which satisfy

$$[x_i, x_j] = \epsilon_{ijk}x_k \quad \text{or indeed} \quad [J_i, J_j] = \mathsf{i}\epsilon_{ijk}J_k \tag{3.4}$$

where $J_k = \mathsf{i}x_k$ for $k = 1, 2, 3$. We will continue with this example later in the section where we take up the task of generating all the real forms of $sl(2,\mathbb{C})$, starting from the compact form $su(2)$.

### 3.1.2 Involutive automorphisms

Although many of the following results apply in a more general setting, we now restrict our attention to the $A_{n-1} \cong sl(n,\mathbb{C})$ series of Lie algebras. This is done to make the exposition as concrete as possible in preparation for the detailed analysis of $su(2,2)$ in final part of this thesis. The compact real form $L = su(n)$ has the following standard basis [33]:

$$\begin{array}{rcll} H^i_i &=& \mathsf{i}(E^i_i - E^{i+1}_{i+1}) & \\ M^i_j &=& E^i_j - E^j_i & (= -M^j_i) \\ N^i_j &=& \mathsf{i}(E^i_j + E^j_i) & (= N^j_i) \end{array} \tag{3.5}$$

where $i, j = 1, .., n$. These generators satisfy the following commutation relations:

$$\begin{array}{rcl} [H^l_l, M^j_k] &=& (\delta^j_l - \delta^l_k - \delta^j_{l+1} + \delta^{l+1}_k)N^j_k \\ [M^j_k, N^l_m] &=& \delta^l_k N^j_m - \delta^j_m N^l_k + \delta^m_k N^j_l - \delta^j_l N^m_k \\ [N^j_k, H^l_l] &=& (\delta^j_l - \delta^l_k - \delta^j_{l+1} + \delta^{l+1}_k)M^j_k \end{array} \tag{3.6}$$

and we note the special case $[M^i_j, N^j_i] = N^i_i - N^j_j = 2\sum\limits_{k=i}^{j-1} H_k$.

An *involutive automorphism* of $L$ is an *inner* automorphism $\sigma : L \to L$ defined by $X \mapsto \sigma X \sigma^{-1}$ and satisfying $\sigma^2 = I$. Every real form of $sl(2,\mathbb{C})$ may be obtained via the action of an involutive automorphism as follows. Since $\sigma^2 = I$ it follows that eigenvalues of $\sigma$ are $\pm 1$. Thus the generators of $L$ decompose into positive and negative eigenspaces denoted by $T$ and $P$ respectively such that:

$$\sigma(T) \to T \qquad \text{and} \qquad \sigma(P) \to -P$$

and $L = T \oplus P$. Furthermore we have

$$[T,T] \subseteq T \qquad [T,P] = P \qquad [P,P] \subseteq T.$$

The real form $L^*$ associated with each $\sigma$ (in this case $L^* = sl(n,\mathbb{R})$ or $L^* = su(p,q)$) is given by

$$L^* = T \oplus \mathsf{i}P$$

The factor of $\mathsf{i}$ converts compact generators in the subspace $P$ of $L$ into noncompact generators in $L^*$.

Real forms of any complex simple Lie algebra may be uniquely identified by their *character* $\zeta(L^*)$ equal to the number of noncompact generators minus the number of compact generators. The compact real form has the character $\zeta(su(n)) = -dim(L^*) = 1 - n^2$.

When $\sigma$ is the operation of complex conjugation then one obtains the least compact (*split*) real form $sl(n,\mathbb{R})$. In this case the negative eigenspace $P \subset L$ consists of precisely those generators in (3.5) that contain a factor of $\mathsf{i}$. The mapping $P \mapsto \mathsf{i}P$ removes this factor and $sl(n,\mathbb{R})$ is simply the real Lie algebra formed by taking the real span of the standard Chevalley or Cartan-Weyl generators. The character of the split real form is equal to the rank $n-1$.

The non-compact real forms $su(p,q)$, where $p+q=n$ and $p,q\neq 0$, are obtained via automorphisms of the form [33]

$$\sigma = \left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array}\right) \tag{3.7}$$

where $I_k$ denotes the identity matrix of dimension $k$. We determine a basis for $su(p,q)$ in the section below.

### 3.1.3 Block Matrix Decomposition of $su(p,q)$

Let $X \in su(n)$ then $X = aH^i_i + bM^i_j + cN^i_j$ and $X^\dagger = -X$. Under the action of $\sigma$ we have, in the fundamental representation, the following block matrix decomposition [33]:

$$\sigma X \sigma^{-1} = \left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array}\right) \left(\begin{array}{c|c} A & B \\ \hline -B^\dagger & C \end{array}\right) \left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array}\right) = \left(\begin{array}{c|c} A & -B \\ \hline B^\dagger & C \end{array}\right) \tag{3.8}$$

where $A^\dagger = -A$ and $C^\dagger = -C$. We may then identify the eigenspaces

$$T = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & C \end{array}\right) \text{ and } P = \left(\begin{array}{c|c} 0 & B \\ \hline -B^\dagger & 0 \end{array}\right). \tag{3.9}$$

Thus $L^* = T \oplus iP \cong su(p,q)$. We define the pseudo-unitary metric

$$\eta = \mathrm{diag}(\underbrace{1,..,1}_{p-terms}, \underbrace{-1,..,-1}_{q-terms})$$

where again $p=n$ and $q=0$ corresponds to the special case $su(n)$. In terms of the standard basis, the eigenspaces $T$ and $P$ are given by

$$\begin{array}{l} T = \mathrm{span}\{M^i_j,\, N^i_j \,|\, \eta^i{}_i \eta^j{}_j = 1\} \cup \mathrm{span}\{H^i_i \,| i = 1,..,n-1\} \\ P = \mathrm{span}\{M^i_j,\, N^i_j \,|\, \eta^i{}_i \eta^j{}_j = -1\} \end{array} \quad \text{(no sum.)} \tag{3.10}$$

Since $P \to \mathrm{i}P$ under the mapping $su(n) \to su(p,q)$ we have

$$\begin{array}{ll} H^i_i & \mapsto H^i_i \\ M^i_j & \mapsto \sqrt{\eta^i{}_i \eta^j{}_j} M^i_j \\ N^i_j & \mapsto \sqrt{\eta^i{}_i \eta^j{}_j} N^i_j. \end{array} \tag{3.11}$$

Also we have

$$[T,T] \subseteq T \qquad [T,\mathrm{i}P] = \mathrm{i}P \qquad [\mathrm{i}P,\mathrm{i}P] \subseteq T,$$

from which it follows that the generators of $su(p,q)$ satisfy a slightly modified version of the relations (3.6) whereby the terms on the right-hand side of the $[M^j_k, N^l_m]$ bracket acquire different signs depending on their compactness [1]. The character is given by $\zeta(su(p,q)) = 1-(p-q)^2$.

[1] An alternative and somewhat more succinct way to express the defining relations of $su(p,q)$ is to set $L^a{}_b = E^a{}_b - \frac{1}{n}\delta^a{}_b E^{a'}{}_{a'}$ and $T^a{}_b = \eta^a{}_{a'} L^{a'}_{b'} \eta^{b'}{}_b$ which satisfy $[T^a{}_b, T^c{}_d] = \eta^a{}_d T^c{}_b - \eta^c{}_b T^a{}_d$.

**Example: The real forms of $sl(2, \mathbb{C})$**

Let us follow on from the previous example and take as basis for the compact real form $su(2)$ the generators $x_1, x_2, x_3$, satisfying the relations (3.4). As we have seen, the split form $sl(2, \mathbb{R})$ is simply the real span of the Chevalley generators $h, e, f$ satisfying (3.2). Thus it remains to determine a basis and the commutation relations for the real form $su(1,1) \cong so(2,1)$. We have $\eta = diag(1, -1)$ which together with (3.10) gives the basis

$$k_1 = x_1 \quad k_2 = ix_2 \quad k_3 = ix_3. \tag{3.12}$$

These satisfy the well-known commutation relations

$$\begin{aligned} [k_1, k_2] &= [x_1, ix_2] = ix_3 = k_3 \\ [k_2, k_3] &= [ix_2, ix_3] = -x_1 = -k_1 \\ [k_3, k_1] &= [ix_3, x_1] = ix_2 = k_2. \end{aligned} \tag{3.13}$$

We note that the characters $\zeta(su(p,q)) = 1 - (p-q)^2$ and $\zeta(sl(n, \mathbb{R})) = n - 1$ are equal for the case $p = q = 1$ thus we have the isomorphism $su(1,1) \cong sl(n, \mathbb{R}))$. In the following section we will complete this small series of examples based around $sl(2, \mathbb{C})$ and its real forms by investigating aspects of the representation theory.

### 3.1.4 Unitary Representations

So far, having selected Lie algebras as the mathematical entry point for this thesis, we have avoided the topic of Lie groups. It is beyond the scope of these introductory chapters to give anything more than the briefest mention of Lie groups which we do now in order give a clearer context for the representation theory. Typically one introduces the topic by first defining the axioms of a Lie group, the distinguishing one being the continuous and parametrisable nature of the elements. It follows that, for an arbitrary Lie group $G$ and $A \in G$, we may write $A = A(\alpha_1, ..., \alpha_n)$ where $\alpha_i$ are continuous group parameters, $n$ is the dimension of $G$ and the identity is given by $A(0) \equiv A(0, 0, ..., 0) = I$. Next, one defines the so-called *infintesimal group generators*

$$x_i = \left( \frac{\partial A}{\partial \alpha_i} \right)_{\alpha=0}$$

which can be shown to satisfy the axioms of a Lie algebra of dimension $n$.

A representation $\pi$ of a Lie Group $G$ is called a *unitary representation* if for each $A' \in G$, $A = \pi(A')$ is a square matrix satisfying

$$A^\dagger A = AA^\dagger = I$$

which implies $A^\dagger = A^{-1}$. Differentiating the above yields

$$\begin{aligned} 0 &= \frac{\partial}{\partial \alpha_i} \left( AA^\dagger \right)_{\alpha=0} \\ &= \left( \frac{\partial A}{\partial \alpha_i} \right)_{\alpha=0} A^\dagger(0) + A(0) \left( \frac{\partial A^\dagger}{\partial \alpha_i} \right)_{\alpha=0} \\ &= x_i^\dagger + x_i \end{aligned}$$

from which is follows that representative matrices of the corresponding Lie algebra are antihermitian. In quantum mechanics it is customary to set $j = \mathrm{i}x$ which yields hermitian generators for unitary representations and guarantees real eigenvalues for observables. For the present analysis we shall continue to adopt the former convention of using antihermitian operators for Lie algebra elements in unitary representations.

Let us now investigate some properties of unitary representations of the real forms of $sl(n, \mathbb{C})$. As we saw in section 2.1.3, the Chevalley basis can be used to explicitly construct the irreducible representations of complex Lie algebras, beginning with a highest weight vector. We now proceed similarly. Starting with the standard basis (3.11) (derived from the compact basis (3.1)) we note that in a unitary representation the generators satisfy

$$H_i^{i\dagger} = -H_i^i \qquad M_j^{i\dagger} = -M_j^i \qquad N_j^{i\dagger} = -N_j^i.$$

We retrieve the Chevalley basis by taking appropriate complex linear combinations of these generators. The form of these mappings depends on the compactness of the generators

- If ${M^i}_j$ and ${N^i}_j$ are compact then $E^i_j = \frac{1}{2}({M^i}_j - i{N^i}_j)$ and $({E^i}_j)^\dagger = {E^j}_i$
- If ${M^i}_j$ and ${N^i}_j$ are noncompact then $E^i_j = -\frac{i}{2}({M^i}_j - i{N^i}_j)$ and $({E^i}_j)^\dagger = -{E^j}_i$

We may succinctly state the above hermiticity conditions in terms of $\eta$ as

$$({E^i}_j)^\dagger = {\eta^i}_{j'}{E^{j'}}_{i'}{\eta^{i'}}_j. \tag{3.14}$$

Finally we have $H_i = -i{H^i}_i$ satisfying $(H_i)^\dagger = H_i$.
As we shall see below, these conditions affect the eigenvalue spectrum of the universal Casimir element and in turn the characterisation of representations. In particular, the unitary irreducible representations of non-compact real forms are all infinite-dimensional [77].

**Universal Casimirs**

The universal Casimir element (2.21) is given by

$$c = g^{ij}x_i x_j$$

where $g^{ij}$ is the Killing metric (2.1) and $x_i$ is any basis for $L$. For the compact real form $su(n)$, we use the standard basis (3.1) expressed in terms of the Gel'fand generators to evaluate

$$g_{ij} = -4n\delta_{ij} \qquad \text{and} \qquad g^{ij} = -\frac{1}{4n}\delta^{ij}.$$

In the case of the noncompact real forms, the metric remains diagonal; however, the additional factor of i belonging to the noncompact terms changes the sign of the corresponding matrix element in $g_{ij}$. The universal Casimir element for $su(p, q)$ is then

$$c = -\sum_{i=1}^{n-1}(H_i^i)^2 - \sum_{i<j}\left[\eta_i^i\eta_j^j\left((M_j^i)^2 + (N_j^i)^2\right)\right]$$

where a factor of $4n$ has been suppressed. The key difference is that the metric is not negative definite for noncompact real forms.

It is also of interest to determine the Casimir operator in the Chevally basis. In this case the metric is no longer diagonal and takes the form

$$(g_{ij}) = 2n \left(\begin{array}{c|ccc} (A_{ij}) & & & \\ \hline & \begin{matrix}0 & 1\\ 1 & 0\end{matrix} & & \\ & & \begin{matrix}0 & 1\\ 1 & 0\end{matrix} & \\ & & & \ddots \end{array}\right) \tag{3.15}$$

where $(A_{ij})$ is the Cartan matrix. The Casimir element in this basis becomes

$$C = A^{ij} H_i H_j + \sum_{i \neq j} E^i_j E^j_i \tag{3.16}$$

where $(A^{ij})$ is the inverse Cartan matrix and an overall factor of $1/2n$ has been suppressed. The Casimir takes this same form for all of the real forms, however, the hermiticy conditions which apply to the root vectors $E^i_j$ are unique to each form. These play an important role in determining the eigenvalue spectrum of $C$ as demonstrated in the low dimensional example below.

**Example: Unirreps of $su(2)$ and $su(1,1)$**

We take (3.3) and (3.12) as a basis for $su(2)$ and $su(1,1)$ respectively. In a unitary representation we have

$$x_i^\dagger = -x_i \quad \text{and} \quad k_i^\dagger = -k_i \quad i = 1, 2, 3.$$

Let us note the following basis transformations of each real Lie algebra to the Chevalley basis

$$\begin{array}{llll} (h) & x_0 = -2ix_1 & (h) & k_0 = -2ik_1 \\ (e) & x_+ = -x_3 - ix_2 & (e) & k_+ = ik_3 - k_2 \\ (f) & x_- = x_3 - ix_2 & (f) & k_- = -ik_3 - k_2 \end{array} \tag{3.17}$$

where in both cases $h, e, f$ satisfy the relations (3.2). In each of these bases the Casimir invariant is

$$c = \frac{1}{2}{x_0}^2 + x_+ x_- + x_- x_+ \quad \text{and} \quad c = \frac{1}{2}{k_0}^2 + k_+ k_- + k_- k_+$$

where a factor of $\frac{1}{4}$ has been suppressed. These satisfy the following h ermiticity conditions:

$$\begin{array}{lll} x_+^\dagger = -x_3^\dagger + ix_2^\dagger = x_3 - ix_2 = x_- & \text{and} & x_-^\dagger = x_+ \\ k_+^\dagger = -ik_3^\dagger - k_2^\dagger = ik_3 + k_2 = -k_- & \text{and} & k_-^\dagger = -k_+. \end{array}$$

Under the basis transformation (3.17), the generators of both real forms satisfy the same set of commutation relations, however, in the case of unitary representations, each satisfies different hermiticity conditions. Consequently, the quadratic Casimir invariant for the compact real form $su(2)$ is real and positive definite but, for the noncompact form $su(1,1)$, it is only real. This leads to the following classification of irreducible unitary representations (for a detailed exposition see p.143-149 of [77]).

$\mathbf{so(3)}$: The real and positive definite eigenvalues of $c$ result in all unitary irreducible representations being finite-dimensional. Each representation is characterised by the label $j = \frac{n}{2}$, $n = 1, 2, ..$ and has dimension $2j+1$. Basis vectors within an irreducible representation are uniquely labeled by the eigenvalues $m$ of the compact generator $h = x_0$. The $m = j$ case corresponds to the maximal weight vector and we may write basis vectors in the form $|jm\rangle$ satisfying

$$x_0|jm\rangle = 2m|jm\rangle \qquad \text{and} \qquad c|jm\rangle = \chi_j(c)|jm\rangle = 2j(j+1)|jm\rangle$$

where $\chi_j$ denotes the central character and $m = -j, -j+1, -j+2, ..., j-2, j-1, j$ enumerates the $2j+1$ states.

$\mathbf{su(1,1) \cong so(2,1) \cong sl(2,\mathbb{R})}$: As a result of $c$ not being positive definite, unitary irreducible representations of this noncompact form are all infinite-dimensional. They come in four distinct series, two of which are continuous and two discrete. The two continuous series are unbounded and characterised by a positive real eigenvalue of $C$, the primary distinction between these two series being the relative shifts in the discrete eigenvalue spectrum of the compact generator. The continuous series are not treated further in this thesis.

The two discrete series are characterised by $k = \frac{n}{2}$, $n = 1, 2, ..$ similar to the compact case. Basis vectors of the infinite-dimensional representations are labeled uniquely by the discrete eigenvalues $m$ of the compact generator $k_0$ and take the form $|km\rangle$. In each case the character is given by

$$c|km\rangle = \pi_k(c)|km\rangle = 2k(k-1)|km\rangle$$

and we have the following:

- The positive discrete series $\pi_k^+$ is bounded below with the lowest weight vector corresponding to $m = k$. We may write the basis vectors as

$$|k, m\rangle \qquad m = k, k+1, k+2, ...$$

- The negative discrete series $\pi_k^-$ is bounded above with the highest weight vector corresponding to $m = -k$. We may write the basis vectors as

$$|k, m\rangle \qquad m = -k, -(k+1), -(k+2), ...$$

## 3.2 Characteristic Identities

This section begins with a worked example of the Cayley-Hamilton theorem followed by an introduction to the general theory of characteristic identities for semisimple Lie algebras. Identities of this form were first introduced by Bracken and Green [9][39] for the Lie algebras $gl(n)$, $so(n)$, $o(n)$ and $sp(n)$. These ideas and techniques were extended by Gould [36] to include the finite-dimensional representations of all semisimple Lie algebras. Finally, Gould [35], building upon the work of Kostant [51], generalised the theory to include infinite-dimensional representations.

The theory of characteristic identities plays an important role in this thesis both as a technique for exposing the decomposition of atypical modules (see §2.2.5 and §5.2) and as a key structural component of the defining relations for certain classes of quadratic Lie superalgebras (see §5.3 and §6.2). These applications will require the most general (infinite-dimensional) form of the theory and as such the presentation below closely follows [35].

### 3.2.1 Cayley-Hamilton Theorem

Let $A$ be an $n \times n$ matrix over a commutative ring. The *characteristic polynomial* $p(\lambda)$ of $A$ is defined to be

$$p(\lambda) = det(\lambda I - A) = \prod (\lambda - a_i),$$

where $I$ is the $n \times n$ identity matrix and $a_i$ are the eigenvalues of $A$. Setting $p(\lambda) = 0$ defines the *characteristic equation* and solving for $\lambda$ yields, of course, the eigenvalues of the matrix $A$. The *Cayley-Hamilton theorem* states that the matrix $A$ satisfies its own characteristic equation; that is

$$p(A) = 0.$$

**Simple example:** Take

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}.$$

Then

$$p(\lambda) = det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 1 \\ 2 & \lambda - 2 \end{vmatrix} = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4).$$

Now

$$p(A) = A^2 - 5A + 4 = \begin{pmatrix} 11 & 5 \\ 10 & 6 \end{pmatrix} - \begin{pmatrix} 15 & 5 \\ 10 & 10 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

### 3.2.2 Characteristic identities of semi-simple Lie algberas

In analogy with the Cayley-Hamilton theorem, the general theory of characteristic identities for a semi-simple Lie algebra $L$ is based around the identification of a (Lie algebra-valued) matrix $A = (A_{ij})$ where $A_{ij} \in U(L)$. The construction of $A$ begins by fixing a finite-dimensional irreducible representation $\pi_\varphi$ of $L$, together with a fixed central term $z \in U(L)$.

$\pi_\varphi$ is called the *reference representation* and is naturally associated with the module $V(\varphi)$. Let $k = dim\, V(\varphi)$. Following Gould [36], we consider the map

$$\partial : U(L) \to [EndV(\varphi)] \otimes U(L)$$

defined for $x \in L$ by

$$\partial(x) = \pi_\varphi(x) \otimes 1 + 1 \otimes x. \tag{3.18}$$

This is extended to an algebra homomorphism to all of $U(L)$. For example taking $x, y \in L$ we have

$$\begin{aligned} \partial(xy) =& (\pi_\varphi(x) \otimes 1 + 1 \otimes x)(\pi_\varphi(y) \otimes 1 + 1 \otimes y) \\ =& \pi_\varphi(x)\pi_\varphi(y) \otimes 1 + \pi_\varphi(x) \otimes y + \pi_\varphi(y) \otimes x + 1 \otimes xy. \end{aligned} \tag{3.19}$$

We now define the matrix

$$\tilde{A} \equiv \tilde{A}_\varphi(z) = -\frac{1}{2}\left[\partial(z) - \pi_\varphi(z) \otimes 1 - 1 \otimes z\right] \tag{3.20}$$

which belongs to $[EndV(\varphi)] \otimes U(L)$.

The aim of the general theory is to determine the polynomial identities satisfied by $\tilde{A}$. These polynomial identities $P(\tilde{A}) = 0$ depend on both the reference representation and the fixed central term $z$. We proceed concretely and let $V(\mu)$ (*resp.* $\pi_\mu$) be any, possibly infinite-dimensional, module (*resp.* representation) of $L$ that admits an infinitesimal character. Now, the Lie algebra valued matrix $\tilde{A}$ will become an ordinary numerical matrix if each element $A_{ij} \in U(L)$ is mapped to $\pi_\mu(A_{ij})$. We define such a matrix

$$A \equiv A^\mu_\varphi(z) = -\frac{1}{2}\left[(\pi_\varphi \otimes \pi_\mu)(z) - \pi_\varphi(z) \otimes 1 - 1 \otimes \pi_\mu(z)\right] \tag{3.21}$$

which belongs to $[EndV(\varphi)] \otimes [EndV(\mu)]$. $A$ satisfies a characteristic identity of degree $dimV(\varphi) \times dimV(\mu)$. Let $\{\varphi_i, .., \varphi_k\}$ be the set of distinct weights of $V(\varphi)$. The eigenvalues, which shall also call the roots, of $A$ may be uniquely determined by the infinitesimal characters occurring in the decomposition of $V(\varphi) \otimes V(\mu)$. It has been shown in [51] that these characters take the form $\mu + \varphi_i$, and as a result the eigenvalues of $A$ may then be written as [36]

$$a_i = -\frac{1}{2}\left[\chi_{\mu+\varphi_i}(z) - \chi_\mu(z) - \chi_\varphi(z)\right] \qquad i = 1, .., k. \tag{3.22}$$

It follows from the Cayley-Hamilton theorem $A$ satisfies the characteristic identity

$$p(A) = \prod_{i=1}^{k}(A - a_i) = 0. \tag{3.23}$$

### 3.2.3 Minimal polynomial identities of semi-simple Lie algebras

In general it is possible that $A$ satisfies a polynomial identity of degree less than $k$. We define the *minimal polynomial identity* $m(A) = 0$ to be the unique polynomial identity of

$A$ having least degree and with the highest coefficient 1. The minimal polynomial identity factors every polynomial identity of $A$ [31].

To determine $m(A)$, let us begin with the characteristic identity (3.23) whose roots are given by (3.22). Although the weights $\varphi_1, ..., \varphi_k$ are all distinct it is not necessarily true that the infinitesimal characters $\chi_{\mu+\varphi_1}, .., \chi_{\mu+\varphi_k}$ are also distinct. Let $n \leq k$ be the number of distinct roots and let us assume that the weights $\varphi_1, ..., \varphi_k$ are numbered such that $\chi_{\mu+\varphi_1}, .., \chi_{\mu+\varphi_n}$ are all distinct. Let $m_i$ be the multiplicity of each infinitesimal character $\chi_{\mu+\varphi_i}$, $i = 1, ..., n$, amongst the set $\chi_{\mu+\varphi_i}$, $i = 1, ..., k$. The characteristic identity (3.23) may then be written

$$p(A) = \prod_{i=1}^{n} (A - a_i)^{m_i} = 0. \tag{3.24}$$

Now $A$, and hence $p(A)$, act on the tensor product module

$$Y \equiv V(\varphi) \otimes V(\mu) \tag{3.25}$$

where we assume that $V(\varphi)$ is finite-dimensional (although not necessarily unitary) and, as before, $V(\mu)$ is any representation admitting a character. Following Kostant [51], $Y$ admits the decomposition

$$Y = \bigoplus_{i=1}^{n} Y_i \tag{3.26}$$

where

$$Y_i = \{y \in Y | (A - a_i)^{m_i} y = 0\}. \tag{3.27}$$

In other words, $Y$ decomposes into a direct sum of $n$ generalised eigenspaces $Y_i$, each one associated the distinct (generalised) eigenvalue $a_i$. The simplest case is when $m_i = 1$, $i = 1, ..., k$, in which case all the infinitesimal characters are distinct and all the $Y_i$ are ordinary eigenspaces. In the general case an element $y \in Y_i$ satisfying $(A - a_i)^r y = 0$ with $(A - a_i)^{r-1} y \neq 0$ is called a *generalised eigenvector* of *rank* $r$.

Following Gould [35], each $L$ submodule $Y_i$ has associated with it a *characteristic length* $n_i$ which is equal to the highest ranked generalised eigenvector belonging to $Y_i$. In other words, although $Y_i$, as defined in (3.27), may possess generalised eigenvectors up to maximal rank $m_i$, the highest ranked generalised eigenvector may in fact be less than this allowing the module $Y_i$ to be more precisely written

$$Y_i = \{y \in Y | (A - a_i)^{n_i} y = 0\} \qquad 0 \leq n_i \leq m_i. \tag{3.28}$$

The special case $n_i = 0$ implies that $Y_i = (0)$.

Using the definitions above, we may now determine the minimal polynomial identity. Starting with the characteristic identity $p(A) = 0$, a polynomial identity of reduced degree exists whenever $n_i < m_i$ for one or more $i \in 1, ..., n$. In this case the factors $(A - a_i)^{m_i}$ of $p(A)$ may be replaced with $(A - a_i)^{n_i}$ lowering the polynomial degree by $\sum_i m_i - n_i$. This follows directly from (3.28) and (3.26) (consider the action of the reduced polynomial on an arbitrary homogenous element $y \in Y_i$). It follows that we may express the minimal polynomial identity satisfied by $A$ as

$$m(A) = \prod_{i=1}^{n} (A - a_i)^{n_i}. \tag{3.29}$$

We see from this that knowledge of the minimal polynomial identity satisfied by $A$ is equivalent to knowledge of the characteristic lengths of the modules $Y_i$ [35].

In the case that $V(\varphi)$ is unitary and $V(\mu)$ is finite-dimensional the situation is greatly simplified. Due to complete reducibility, each submodule $Y_i$ is an ordinary eigenspace of $A$ (that is $n_i \leq 1$ even in the case of repeated roots). Furthermore, we have $Y_i = (0)$ $(n_i = 0)$ whenever $\mu + \varphi \notin D^+$ ($\mu + \varphi_i$ is not dominant integral). The minimal polynomial identity, in this case, is simply the degree $n$ polynomial [36]

$$m(A) = \prod_{\substack{i=1 \\ \mu+\varphi_i \in D^+}}^{n} (A - a_i) \qquad \text{(when } V(\mu) \text{ is finite-dimensional.)} \tag{3.30}$$

Finally, returning to the general case, we see that the precise form of (3.29) depends on the representation $\pi_\mu$. In particular, the satisfaction of a polynomial identity of reduced degree places constraints on $\mu$.

It is, of course, possible to expand any factored polynomial identity of $A$ to obtain a polynomial expression whose coefficients are a function of the eigenvalues $a_i$. It turns out that each of these coefficients are the eigenvalues of central elements belonging to $U(L)$ [35]. This enables the polynomial identity to be expressed in a representation independent form in terms of the abstract matrix $\tilde{A}$. In particular, the minimal polynomial identity (3.29), with degree $q = \sum n_i$, may be written as

$$p(\tilde{A}) = C_0(z) + C_1(z)\tilde{A} + C_2(z)\tilde{A}^2 + ... + C_p(z)\tilde{A}^q = 0$$

where $C_i(z) \in Z$ depend only on the fixed central element $z$ and the fixed (reference) weight $\varphi$.

For notational convenience we will often identify $\tilde{A}$ with $A$ where it is understood that the abstract matrix $\tilde{A}$ assumes the numerical form $A$ whenever the representation $V(\mu)$ is fixed.

### 3.2.4 Low Dimensional Examples

In the following examples we fix $z = c$ to be the universal (quadratic) Casimir (2.21). In addition, we fix the reference representation $\pi_\varphi$ to be the fundamental contragredient representation (recall that $\varphi = \omega_1^*$ is the negative of lowest weight of the fundamental representation $\pi_{\omega_1}$). These particular choices correspond to the simplest possible cases and they generate the well-known results derived below. Let us begin by investigating

$$\tilde{A} = \tilde{A}_\varphi(c) = -\frac{1}{2}\left[\partial(c) - \pi_\varphi(c) \otimes 1 - 1 \otimes c\right].$$

Using (3.19) we have (with summation over $i$)

$$\begin{aligned}\partial(c) = \partial(x^i x_i) &= \pi_\varphi(x^i)\pi_\varphi(x_i) \otimes 1 + \pi_\varphi(x^i) \otimes x_i + \pi_\varphi(x_i) \otimes x^i + 1 \otimes x^i x_i \\ &= \pi_\varphi(x^i) \otimes x_i + \pi_\varphi(x_i) \otimes x^i + \pi_\varphi(c) \otimes 1 + 1 \otimes c\end{aligned}$$

which gives

$$\tilde{A} = -\frac{1}{2}\left[\pi_\varphi(x^i)\otimes x_i + \pi_\varphi(x_i)\otimes x^i\right]. \tag{3.31}$$

We now apply this result to the low-dimensional examples of $gl(2)$ and $sl(2)$. We shall take up the investigation of the general cases $gl(n)$ and $sl(n)$ later in the chapter.

### A characteristic identity of $gl(2)$

We begin by explicitly calculating $\tilde{A}$. Matrices representing the basis vectors ${E^i}_j$ $i,j = 1,2$ in the fundamental contragredient representation are by definition the negative transpose of those in the fundamental representation. Thus we have

$$\pi_\varphi({E^i}_j) = -{e^j}_i.$$

Using (3.31), where $c = x^i x_i = {E^i}_j {E^j}_i$ for $i,j = 1,2$, we have (summing over $i,j$ on each line)

$$\begin{aligned} E \equiv \tilde{A} &= -\frac{1}{2}\left[\pi_\varphi({E^i}_j)\otimes {E^j}_i + \pi_\varphi({E^j}_i)\otimes {E^i}_j\right] \\ &= -\frac{1}{2}\left[-{e^j}_i\otimes {E^j}_i - {e^i}_j\otimes {E^i}_j\right] \\ &= {e^i}_j\otimes {E^i}_j \\ &= \begin{pmatrix} {E^1}_1 & {E^1}_2 \\ {E^2}_1 & {E^2}_2 \end{pmatrix}. \end{aligned} \tag{3.32}$$

Next we calculate the eigenvalues

$$a_i = -\frac{1}{2}\left[\chi_{\mu+\varphi_i}(c_2) - \chi_\mu(c_2) - \chi_\varphi(c_2)\right]. \tag{3.33}$$

The two distinct weights of $\pi_\varphi$ (the fundamental contragredient) are

$$\varphi_1 = -\varepsilon_2 \quad (=\varphi) \tag{3.34}$$

$$\varphi_2 = -\varepsilon_2 - \alpha = -\varepsilon_1. \tag{3.35}$$

We let

$$\mu = \mu_1\varepsilon_1 + \mu_2\varepsilon_2$$

be an arbitrary dominant integral weight such that $|\mu\rangle \in V(\mu)$ is a highest weight vector satisfying

$$ {E^i}_i|\mu\rangle = \mu_i|\mu\rangle.$$

$gl(2)$ has both a linear and quadratic Casimir operator which we label $c_1$ and $c_2$ respectively. In our chosen basis we have

$$\begin{aligned} c_1 &= E_1^1 + E_2^2 & \chi_\mu(c_1) &= \mu_1 + \mu_2 \\ c_2 &= (E_1^1)^2 + (E_2^2)^2 + E_1^1 - E_2^2 + 2E_1^2E_2^1 & \chi_\mu(c_2) &= (\mu_1)^2 + (\mu_2)^2 + \mu_1 - \mu_2 \end{aligned} \tag{3.36}$$

such that $c_i|\mu\rangle = \chi_\mu(c_i)|\mu\rangle$. (We note that $\chi_\mu(c_2)$ could also be evaluated using (2.22)). Employing (3.36), we have

$$\begin{aligned}\chi_\varphi(c_2) &= (-1)^2 - (-1) = 2\\ \chi_{\mu+\varphi_1}(c_2) &= (\mu_1)^2 + (\mu_2-1)^2 + \mu_1 - (\mu_2 - 1)\\ \chi_{\mu+\varphi_2}(c_2) &= (\mu_1-1)^2 + (\mu_2)^2 + (\mu_1 - 1) - \mu_2\end{aligned}$$

which we insert in (3.33) to obtain

$$\begin{aligned}a_1 &= \mu_2\\ a_2 &= \mu_1 + 1.\end{aligned}$$

Using (3.23) the characteristic identity is $p(E) = (E - \mu_2)(E - \mu_1 - 1)$. This expands to

$$p(E) = E^2 - (\mu_1 + \mu_2 + 1)E + \mu_1\mu_2 + \mu_2 = 0.$$

Using (3.36) it is straight forward to show that

$$p(E) = E^2 - (c_1 + 1)E + \frac{1}{2}(c_1^2 + c_1 - c_2) = 0 \tag{3.37}$$

which is identical to Gould (pg.13 [36].) The matrix identity (3.37) may be written in index form as

$$E^i{}_k E^k{}_j - (c_1+1)E^i{}_j + \frac{1}{2}\delta^i{}_j(c_1^2 + c_1 - c_2) = 0 \qquad\qquad i, j, = 1, 2$$

We note that (3.37) is also the mimimal polynomial identity of $E$ as the assumption that $\mu$ is dominant integral prohibits the possibility of coincident roots; for $a_1 = a_2$ requires $\mu_2 = \mu_1 + 1$, however, by assumption $\mu_1 - \mu_2 \in \mathbb{Z}^+$.

### A characteristic identity of $sl(2)$

We easily obtain the characteristic identities for $sl(2)$ from those of $gl(2)$ by simply setting the linear Casimir $c_1 = 0$. Explicitly

$$p(A) = A^2 - A - \frac{1}{2}c_2 \tag{3.38}$$

where, using (3.31),

$$\begin{aligned} A &= -\frac{1}{2}\left[\pi_\lambda(h)\otimes h + 2\pi_\lambda(e)\otimes f + 2\pi_\lambda(f)\otimes e\right]\\ &= \begin{pmatrix} \frac{1}{2}\pi_\mu(h) & \pi_\mu(e)\\ \pi_\mu(f) & -\frac{1}{2}\pi_\mu(h)\end{pmatrix}. \end{aligned} \tag{3.39}$$

Let us follow the general prescription to verify (3.38). As before we set

$$\varphi = \varphi_1 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2) \qquad \varphi_2 = -\frac{1}{2}(\varepsilon_1 - \varepsilon_2) \qquad \text{and} \qquad \mu = j(\varepsilon_1 - \varepsilon_2).$$

We write the quadratic Casimir operator and its eigenvalue as

$$c_2 = \frac{1}{2}h^2 + h + 2fe \qquad \text{and} \qquad \chi_\mu(c_2) = 2j(j+1).$$

It remains to calculate the eigenvalues $a_i$. Using the above and (3.33) we obtain

$$\begin{aligned} a_1 &= -j \\ a_2 &= 1 + j. \end{aligned}$$

Now $p(A) = (A+j)(A-j-1) = A^2 - A - j(j+1) = 0$ which is clearly equivalent to (3.38).

### 3.2.5 Lowest Weight Modules $V(\mu)$

An important case, which concerns in particular the unitary irreducible representations of real forms, is when the representation $V(\mu)$ is characterised by a lowest weight rather than a highest weight. Let us re-examine the rule (3.22) for calculating the eigenvalues $a_i$.

The calculation depends entirely on the central character $\chi_\lambda : Z \to \mathbb{C}$ which maps the fixed central term $z$ to its eigenvalue on a representation characterised by the weight $\lambda$. In the standard case, where $\lambda$ is a highest weight and $z$ is universal Casimir (2.21), the eigenvalue is given by the well-known rule (2.22) which we rewrite here for convenience

$$\chi_\lambda(c) = \langle \lambda, \lambda + 2\rho \rangle.$$

The derivation of this is a textbook procedure (see for example p.239 [15] or p.121 [41]) achieved by evaluating the action of

$$c = x^i x_j = h^i h_i + \sum_{\alpha \in \Delta^+} (e_\alpha f_\alpha + f_\alpha e_\alpha)$$

on a highest weight vector $v_\lambda$. The action of the first term in the expression is

$$h^i h_i . v_\lambda = \langle \lambda, \lambda \rangle v_\lambda.$$

The second term can be rearranged in one of two ways using the commuation relation $[e_\alpha, f_\alpha] = e_\alpha f_\alpha - f_\alpha e_\alpha = h_\alpha$ to give

$$\sum_{\alpha \in \Delta^+} (e_\alpha f_\alpha + f_\alpha e_\alpha) = \sum_{\alpha \in \Delta^+} (h_\alpha + 2 f_\alpha e_\alpha) \tag{3.40a}$$

$$= \sum_{\alpha \in \Delta^+} (-h_\alpha + 2 e_\alpha f_\alpha). \tag{3.40b}$$

Employing the first rearrangement, (3.40a) acts on the highest weight vector $v_\lambda$

$$\sum_{\alpha \in \Delta^+} (h_\alpha + 2 f_\alpha e_\alpha) . v_\lambda = 2\langle \lambda, \rho \rangle v_\lambda$$

where $\rho$ is given by (2.23). The standard result follows.

We now consider the case where the representation is characterised by a *lowest weight*. Let $\mu$ be the lowest weight of a module $V(\mu)$. The eigenvalue $\chi_\mu(c)$ is determined by the action of (3.40b) on the corresponding lowest weight vector $v_\mu$

$$\sum_{\alpha\in\Delta^+} \left(-h_\alpha + 2e_\alpha f_\alpha\right).v_\mu = -2\langle\mu,\rho\rangle v_\mu.$$

Combining the line above with $h^i h_i.v_\mu = \langle\mu,\mu\rangle v_\mu$ yields

$$\chi_\mu(c) = \langle\mu, \mu - 2\rho\rangle \qquad \text{(when } \mu \text{ a lowest weight vector.)} \tag{3.41}$$

Finally, in order to use (3.41) as a means to evaluate (3.22) we must characterise both $V(\mu)$ and $V(\varphi)$ (the finite-dimensional reference representation) in terms of lowest weights. This allows the weights $\mu + \varphi_i$ $i = 1, ..., k$ belonging to the weight space of tensor module (3.25) to be identified as candidate lowest weights in the decomposition (3.26). See below for examples of the characteristic identities arising from both highest weight and lowest weight representations.

### 3.2.6 Example

The following example concerns the Lie algebra $gl(n)$ and has been chosen both to illustrate the general theory of characteristic identities as well to develop results that will be useful in future chapters. We defer calculations of the minimal polynomial identity, particularly in the case of infinite-dimensional representations, until the final chapter where we investigate unitary representations of the real form $u(2,2)$.

Take $L = gl(n)$, and fix $V(\varphi)$ to be the fundamental contragredient representation. It is straight-forward to see that the calculations preceding (3.32) for $gl(2)$ generalise immediately to $gl(n)$, such that for $z = c = E^i{}_j E^j{}_i$

$$\begin{aligned} E \equiv \tilde{A} &= -\frac{1}{2}\left[\pi_\varphi(E^i{}_j) \otimes E^j{}_i + \pi_\varphi(E^j{}_i) \otimes E^i{}_j\right] \\ &= e^i{}_j \otimes E^i{}_j \end{aligned}$$

is simply the $n \times n$-matrix for which the $(i,j)^{th}$-element is the $gl(n)$ generator $E^i{}_j$.

Owing to the reduction $gl(n) = sl(n) \oplus gl(1)$, the Lie algebra $gl(n)$ has the same root system as $sl(n) \cong A_{n-1}$ (see (2.9)). Thus, using (2.11) and (2.23), we have

$$\langle\rho, \alpha_i\rangle = \frac{\langle\alpha_i, \alpha_i\rangle}{2} = 1. \tag{3.42}$$

This result will be useful for the determination of the roots of characteristic identities for $gl(n)$. We shall treat separately the two cases where $V(\mu)$ is either a lowest weight or alternatively a highest weight representation of $gl(n)$.

Case 1: $\mu$ is the **highest weight** of $V(\mu)$.

We begin by rewriting (3.22) in terms of the character rule (2.22) to give

$$\begin{aligned} a_i &= -\frac{1}{2}\left(\langle \mu+\varphi_i, \mu+\varphi_i+2\rho\rangle - \langle \mu, \mu+2\rho\rangle - \langle \varphi, \varphi+2\rho\rangle\right) \\ &= \frac{1}{2}\langle \varphi, \varphi\rangle - \frac{1}{2}\langle \varphi_i, \varphi_i\rangle + \langle \varphi-\varphi_i, \rho\rangle - \langle \varphi_i, \mu\rangle. \end{aligned} \tag{3.43}$$

The highest weight of the fundamental contragredient representation is

$$\varphi = -\varepsilon_n. \tag{3.44}$$

Let us write the $i^{th}$-weight of $V(\varphi)$ as

$$\varphi_i = \varphi - \sum_{j=1}^{i-1} \alpha_i = -\varepsilon_{n+1-i}. \tag{3.45}$$

It follows that $\langle \varphi, \varphi\rangle = \langle \varphi_i, \varphi_i\rangle = 2$ and (3.43) simplifies to

$$a_i = \langle \varphi - \varphi_i, \rho\rangle - \langle \varphi_i, \mu\rangle. \tag{3.46}$$

Let us write $\mu = \sum_{j=1}^{n} \mu_j \varepsilon_j$ then, using (3.42), together with (3.45), gives

$$\langle \varphi - \varphi_i, \rho\rangle = \langle \sum_{j=1}^{i-1} \alpha_i, \rho\rangle = i-1$$

and

$$\langle \varphi_i, \mu\rangle = \langle -\varepsilon_{n+1-i}, \sum_{j=1}^{n} \mu_j \varepsilon_j\rangle = -\mu_{n+1-i}.$$

Substituting the two equations above into (3.46) yields $a_i = \mu_{n+1-i} + i - 1$. We observe, however, that by reordering the roots such that $a_i \to a_{n+1-i}$ we obtain the simpler form found in the literature [60]

$$a_i = \mu_i + n - i \qquad (\mu \text{ is a highest weight}). \tag{3.47}$$

Case 2: $\mu$ is the **lowest weight** of $V(\mu)$.

Again we begin by rewriting (3.22), however, this time we use the character rule (3.41) to give

$$\begin{aligned} a_i &= -\frac{1}{2}\left(\langle \mu+\varphi_i, \mu+\varphi_i-2\rho\rangle - \langle \mu, \mu-2\rho\rangle - \langle \varphi, \varphi-2\rho\rangle\right) \\ &= \frac{1}{2}\langle \varphi, \varphi\rangle - \frac{1}{2}\langle \varphi_i, \varphi_i\rangle + \langle \varphi-\varphi_i, -\rho\rangle - \langle \varphi_i, \mu\rangle. \end{aligned} \tag{3.48}$$

The lowest weight of the fundamental contragredient representation is

$$\varphi = -\varepsilon_1. \tag{3.49}$$

Let us write the $i^{th}$-weight of $V(\varphi)$ as

$$\varphi_i = \varphi + \sum_{j=1}^{i-1} \alpha_i = -\varepsilon_i. \tag{3.50}$$

Again (3.48) may be simplified to

$$a_i = \langle \varphi - \varphi_i, -\rho \rangle - \langle \varphi_i, \mu \rangle. \tag{3.51}$$

As before, we write $\mu = \sum_{j=1}^{n} \mu_j \varepsilon_j$ then, using (3.42), together with (3.50), gives

$$\langle \varphi - \varphi_i, -\rho \rangle = \langle - \sum_{j=1}^{i-1} \alpha_i, -\rho \rangle = i - 1$$

and

$$\langle \varphi_i, \mu \rangle = \langle -\varepsilon_i, \sum_{j=1}^{n} \mu_j \varepsilon_j \rangle = -\mu_i.$$

Finally, we substitute the two equations above into (3.51) obtaining

$$a_i = \mu_i + i - 1 \qquad\qquad (\mu \text{ is a lowest weight}). \tag{3.52}$$

In both cases $E$ satisfies the characteristic identity $\prod_{i=1}^{n}(E - a_i) = 0$. As we have seen in the general theory the characteristic identities for $gl(n)$ may be expressed in the representation independent form

$$c_0 I + c_1 E + c_2 E^2 + ... + c_n E^n = 0$$

where $c_i$ belong to the centre of $U(gl(n))$. A recursive method for the determining coefficients $c_i$ for arbitrary $n$ has been given in [39].

# Chapter 4

# Quadratic Superalgebras

In this chapter we define the notion of a quadratic superalgebra. It is around this key object that the original work in this thesis is based. We defer the presentation of detailed and illustrative examples to the following chapter and instead focus here on the abstract properties of quadratic superalgebras. These include the establishment of the PBW theorem and general results concerning Kac-type modules and conditions for atypicality. In §4.4 we frame the class of quadratic superalgebras within the much larger class of quadratic algebras. In this setting there exists a generalisation of the classical PBW theorem to which there corresponds a set of generalised Jacobi identities, and in certain cases, an algorithm to derive a basis of ordered monomials. We give here a brief account of the main definitions and theorems and then apply these to derive an alternative proof of the PBW Theorem for quadratic superalgebras. In particular, we employ a special case of the so-called Diamond Lemma for non-commutative algebras which will prove to be a useful tool for analysing modified quadratic superalgebras in chapter 5.

The key results in this chapter have been published in a jointly authored paper [46]. Unless otherwise stated, the results presented in this chapter are, in the largest proportion, due to the candidate; in particular the PBW Basis Theorem 4.2.1 and the generalised PBW results of Lemmas 4.4.6, 4.4.7 and 4.4.8.

## 4.1 Formal Definitions

Consider the following generalisation of the structural relations of an ordinary Lie superalgebra

$$[L_{\bar{0}}, L_{\bar{0}}] \subset L_{\bar{0}} \qquad [L_{\bar{0}}, L_{\bar{1}}] \subset L_{\bar{1}} \qquad \{L_{\bar{1}}, L_{\bar{1}}\} \subset L_{\bar{0}} \otimes L_{\bar{0}} + L_{\bar{0}} + \mathbb{C}, \tag{4.1}$$

where $L = L_{\bar{0}} + L_{\bar{1}}$ is a finite-dimensional $\mathbb{Z}_2$-graded complex vector space. The closure of the anticommuator $\{\, , \}$ onto elements of up to (tensor) degree 2 in the underlying vector space may be viewed as a quadratic deformation of the Lie superalgebra structure. Take the even and odd subspaces $L_{\bar{0}}$ and $L_{\bar{1}}$ to be spanned by basis elements $x_1$, $i = 1, 2, ..., n$, and $y_r$, $r = 1, 2, ..., m$, respectively. The tensor algebra $T(L) = \bigoplus_{n=0}^{\infty} \otimes^n(L) \cong \mathbb{C} \oplus L \oplus L \otimes L \oplus \cdots$

inherits the $\mathbb{Z}_2$-grading in the natural way, in that for $T^n := \otimes^n(L)$ we have

$$T^n|_{\bar{0}} \cong \bigoplus_{\sum \bar{\alpha}_i \equiv \bar{0}} L_{\bar{\alpha}_1} \otimes L_{\bar{\alpha}_2} \otimes \cdots \otimes L_{\bar{\alpha}_n}, \qquad T^n|_{\bar{1}} \cong \bigoplus_{\sum \bar{\alpha}_i \equiv \bar{1}} L_{\bar{\alpha}_1} \otimes L_{\bar{\alpha}_2} \otimes \cdots \otimes L_{\bar{\alpha}_n},$$

with each $\bar{\alpha}_i = \bar{0}$ or $\bar{1}$, and $T^n \cong T^n|_{\bar{0}} + T^n|_{\bar{1}}$. Imposing the multiplicative conditions (4.1) we have in terms of basis elements

$$[x_i, x_j] = c_{ij}{}^k x_k \qquad [x_i, y_p] = \bar{c}_{ip}{}^q y_q \qquad \{y_p, y_q\} = d_{pq}{}^{kl} x_k \otimes x_l + b_{pq}{}^k x_k + a_{pq}, \tag{4.2}$$

where the arrays of complex numbers $c_{ij}{}^k$, $\bar{c}_{ip}{}^q$, $b_{pq}{}^k$ and $d_{pq}{}^{kl}$ take the role of generalised structure constants. Demanding that $[\ ,\ ]$ and $\{\ ,\ \}$ fulfill the standard (graded) commutation relations we obtain a set of quadratic relations $I \subset L \otimes L + L + \mathbb{C}$ spanned by the set

$$\begin{array}{l} x_i \otimes x_j - x_j \otimes x_i - c_{ij}{}^k x_k; \\ x_i \otimes y_p - y_p \otimes x_i - \bar{c}_{ip}{}^q y_q; \\ y_p \otimes y_q - y_q \otimes y_p - d_{pq}{}^{kl} x_k \otimes x_l - b_{pq}{}^k x_k - a_{pq}. \end{array} \tag{4.3}$$

Let us demand that the relations (4.2) satisfy the ordinary graded Jacobi identities. Let $\{w_i, i = 1, \cdots, m+n\}$ be a fixed homogeneous basis for $L$ such that $|i| = 0$ and $|i| = 1$ denote the even and odd grading respectively. In this bases the quadratic relations (4.3) take the form

$$w_i w_j - (-1)^{|i||j|} w_j w_i - d_{ij}{}^{lm} w_l w_m - c_{ij}^k w_k - a_{ij}, \tag{4.4}$$

where $d_{ij}{}^{lm}$ and $a_{ij}$ are non-vanishing only when both $|i| = |j| = 1$ and $|l| = |m| = 0$. The graded Jacobi identities, equivalent to (2.43), are

$$[w_i, [w_j, w_k]] = [[w_i, w_j], w_k] + (-1)^{|i|.|j|}[w_j, [w_i, w_k]], \tag{4.5}$$

where nested brackets of the form $[\{y_p, y_q\}, w_i]$, for which the inner bracket contains quadratic terms, are defined by

$$[\{y_p, y_q\}, w_i] \equiv d_{pq}{}^{lo}\left(x_l[x_o, w_i] + [x_l, w_i]x_o\right) + c_{pq}^k[x_k, w_i]. \tag{4.6}$$

In terms of the generalised structure constants the Jacobi identities (4.5) take the form

$$\begin{array}{l} c_{ij}{}^l c_{lk}{}^o = c_{ik}{}^l c_{lj}{}^o + c_{jk}{}^l c_{li}{}^o; \\ c_{ij}{}^l \bar{c}_{lp}{}^q = \bar{c}_{ir}{}^q \bar{c}_{jp}{}^r - \bar{c}_{jr}{}^q \bar{c}_{ip}{}^r; \\ c_{in}{}^k d_{pq}{}^{nl} + c_{in}{}^l d_{pq}{}^{kn} = \bar{c}_{ip}{}^s d_{sq}{}^{kl} + \bar{c}_{ip}{}^s d_{sq}{}^{kl}, \\ b_{pq}{}^o c_{io}{}^n = \bar{c}_{ip}{}^s b_{sq}{}^n + \bar{c}_{iq}{}^s b_{ps}{}^n, \\ \bar{c}_{op}{}^s b_{qr}{}^o + \bar{c}_{oq}{}^s b_{rp}{}^o + \bar{c}_{or}{}^s b_{pq}{}^o = 0, \\ \bar{c}_{op}{}^s d_{qr}{}^{ol} + \bar{c}_{oq}{}^s d_{rp}{}^{ol} + \bar{c}_{or}{}^s d_{pq}{}^{ol} = 0. \end{array} \tag{4.7}$$

**Definition 4.1.1 (Quadratic Superalgebra)**
*Let $L = L_{\bar{0}} + L_{\bar{1}}$ be a finite-dimensional $\mathbb{Z}_2$-graded complex vector space satisfying the multiplicative relations* (4.2) *and the Jacobi identities* (4.7) *above. Let* $(I)$ *be the two-sided ideal in the tensor algebra* $T(L)$ *generated by the set of qudratic relations* $I$ *as in* (4.3)*. The subalgebra of the tensor algebra defined by* $U(L) = T(L)/(I)$ *is called the quadratic superalgebra associated with* $I$*.*

Note when ${d_{pq}}^{kl} = {b_{pq}}^k = 0$ then $U(L) = T(L)/(I)$ is the universal enveloping algebra of the ordinary Lie superalgebra $L = L_{\bar{0}} + L_{\bar{1}}$.

While it is beyond the scope of this work to investigate the enumeration and classification of quadratic superalgebras we can derive, using the notation above, a binary typology analogous to that well known for Lie superalgebras. In the absence of classification theorems, this narrows the class of algebras under study to those possessing certain useful properties.

Due to the presence of nonlinear brackets, there is no natural adjoint action $ad_w : L \to L$. However, from (4.3) we can define for $x \in L_0$ $ad_x : L_{\bar{1}} \to L_{\bar{1}}$ by

$$\mathrm{ad}_x(y) = xy - yx \tag{4.8}$$

for each $y \in L_1$, such that $\mathrm{ad}_{[x,x']} = \mathrm{ad}_x \mathrm{ad}_{x'} - \mathrm{ad}_{x'} \mathrm{ad}_x$. Thus $L_{\bar{1}}$ is a finite-dimensional $L_{\bar{0}}$-module. We impose the extra assumption that $ad_x$ is a representation equivalent to its contragradient and adopt the following nomenclature due to Jarvis [46]:

**Definition 4.1.2 (Balanced quadratic superalgebras)**
*A quadratic superalgebra is called balanced if the odd submodule $L_{\bar{1}}$ is a real representation of the even Lie subalgebra $L_{\bar{0}}$.*

**Definition 4.1.3 (Typology of balanced quadratic superalgebras)**
*Let $\gamma$ be the highest weight of an irreducible representation of $L_0$ with contragredient $\gamma^*$, and $V_0(\gamma)$, $V_0(\gamma^*)$ the corresponding $L_0$-modules. Balanced quadratic superalgebras are categorised into the following types according to the odd submodule $L_{\bar{1}}$:*

*Type I′:* $L_{\bar{1}} = L_+ + L_- \cong V_0(\gamma) + V_0(\gamma^*)$, *for $\gamma$ a complex representation, with* $\{L_\pm, L_\pm\} = 0$;
*or Type II:* $L_{\bar{1}} \cong V_0(\gamma)$, *for $\gamma$ a real representation.*

Note that the extra assumption $\{L_\pm, L_\pm\} = 0$ for Type I′ is stronger than for Type I Lie superalgebras, where the classification theorems are enough to guarantee it.

## 4.2 PBW Basis Theorem

The PBW theorem plays an important role in the development of the representation theory for Lie (super)algebras. In the case of irreducible representations, for example, the availability of an ordered monomial (PBW) basis for the universal enveloping algebra allows the structure of the entire module to be generated from action of basis elements on a highest weight vector.

The starting place of the classical PBW theorem is a total ordering on the basis elements of the underlying ($\mathbb{Z}_2$-graded) vector space $L$. The theorem asserts (see section 2.1.3) that the ordered monomials are a basis for $U(L)$. The standard method of proof relies firstly on an inductive argument to show that the ordered monomials span $U(L)$ and secondly on further inductive arguments, together with fulfillment of the Jacobi identity, to prove their linear independence. In what follows, we adapt the textbook method of Serre [65] (see also

[19]) to prove the existence of a PBW basis for *all* quadratic Lie superalgebras under some mild assumptions on the ordering.

Fix a total ordering on the index set $\mathcal{I} = \{1, 2, \cdots, m+n\}$, where $i \in \mathcal{I}$ labels homogeneous basis elements $w_i \in L$. A *word* of $\mathcal{I}$ is an ordered expression $M = (i_1, i_2, ..., i_k)$, where here and in the following, *ordered* now means that the even (odd) indices are weakly (strictly) increasing. An *ordered monomial* of length $\ell(M) = k$ has the form $w_M = w_{i_1} w_{i_2} \cdots w_{i_k}$. The set of monomials $w_M$ are themselves ordered first by degree (length), then by the order on $\mathcal{I}$, such that $w_{l_1} w_{l_2} ... w_{l_s} > w_{m_1} w_{m_2} ... w_{m_t}$ if $s > t$, or $s = t$ and there exists $v \leq t$ such that $(l_1 = m_1), \ldots, (l_{v-1} = m_{v-1})$ and $l_v > m_v$.

**Theorem 4.2.1 (PBW basis)** *As before let $\{w_i, i = 1, \cdots, m+n\}$ be a fixed homogeneous basis for $L$, and consider the defining relations* (4.4) *which generate the ideal* $(I)$. *Then the quadratic superalgebra $U(L) = T(L)/(I)$ has a basis of ordered monomials if an index ordering exists such that only those ${d_{ij}}^{lo}$ are nonvanishing for which both $l$ and $o$ precede $i$ and $j$.*

**Proof**:
For a fixed total ordering $\mathcal{I} = \{1, 2, \cdots, m+n\}$ assume that both $i$ and $j$ are greater than $l$ and $o$ whenever ${d_{ij}}^{lo}$ is nonzero. Let $w = w_{k_1} w_{k_2} ... w_{k_u}$ be an arbitrary monomial. We assume by induction on the above ordering that any monomial which is less than $w$ may be written as a linear sum of ordered monomials $w_L$ where $\ell(L) \leq u$. If $w$ is not ordered then it can be written in the form $z = a w_{k_p} w_{k_{p+1}} b$ where $a$ and $b$ are monomial expressions and $w_{k_p} \equiv w_j$, $w_{k_{p+1}} \equiv w_i$ with $i < j$. Using the defining relations, we may write

$$w = a(w_i w_j + [w_j, w_i])b = a w_i w_j b + a[w_j, w_i]b.$$

Both of the terms on the right hand side are less than $w$. The first term $a w_i w_j b$ since $i < j$ and the second term $a[w_j, w_i]b$, clearly, if the bracket involves just linear terms, and also for the quadratic case, $|i|, |j| = 1$, since the quadratic part satisfies ${d_{ij}}^{lo} x_l x_o < w_j w_i$. This proves the ordered monomials $w_M$ span $U(L)$, it remains to prove their linear independance.

Let $V$ be the $\mathbb{K}$-vector space with basis $\{z_M\}$ indexed by all admissible words (including $z_\emptyset$ for the empty word). For $M = (a_1, a_2, \cdots, a_m)$ and $a < a_1$ (or $a = a_1$ and $w_a \in L_0$) we shall say that $a \leq M$ and let $aM = (a, a_1, a_2, \cdots, a_m)$. Otherwise we write $a > M$. We show that $V$ can be made into a $U(L)$-module in such a way that $w_a z_M = z_{aM}$ whenever $a \leq M$. Given this, the linear independence of the $w_M$ is easily shown. For it is clear by induction that $w_M z_\emptyset = z_M$, so if $0 = \sum c_M w_M$ where $c_M \in \mathbb{K}$, then $0 = \sum c_M w_M z_\emptyset = \sum c_M z_M$. However, this immediately implies $c_M = 0$, because the $z_M$ are linearly independent by assumption.

The remainder of the proof consists in constructing a suitable module action on $V$, which we do in the selected basis $\{w_i\}$. We need to define $w_i z_M$ for all $i$ and $M$. We may assume by induction that $w_j z_N$ is defined for all $j$ when $\ell(N) < \ell(M)$, and for $j < i$ when $\ell(N) = \ell(M)$. We assume this has been done in such a way that $w_j z_N$ is a $\mathbb{K}$-linear combination of $z_L$'s with $\ell(L) \leq \ell(N) + 1$. We define

$$w_i z_M = \begin{cases} z_{iM}, & \text{if } i < M \\ (-1)^{|i||j|} w_j w_i z_N + [w_i, w_j] z_N, & \text{if } M = jN \text{with } i > j \\ \frac{1}{2}[w_i, w_i] z_N & \text{if } M = iN \text{ and } w_i \in X_1. \end{cases} \tag{4.9}$$

If either $|i| = 0$ or $|j| = 0$ then it is clear from the inductive assumptions that the above are well defined since $d_{ab}{}^{cd} = 0$. For the case $|i| = |j| = 1$ then $(l, o) < (i, j)$ implies that the quadratic term is well defined by inductive assumption also. We need only show that

$$w_i w_j z_N = (-1)^{|i||j|} w_j w_i z_N + [w_i, w_j] z_N \tag{4.10}$$

for all $i, j$ and $N$. Due to the (graded) antisymmetry of (4.10) we may assume $i > j$ and $i \geq j$ for the case $|i| = |j| = 1$. By induction on $\ell(N)$ we may assume that

$$w_i w_j z_L = (-1)^{|i||j|} w_j w_i z_L + [w_i, w_j] z_L \tag{4.11}$$

holds for all $w_i, w_j$ for $\ell(L) < \ell(N)$ (since (4.10) is true for $N = \emptyset$). If $j < N$ then (4.10) follows immediately from (4.9) and we need only consider the case $j \geq N$. Let $N = kL$, thus we have $i \geq j \geq k$, and (4.10) becomes

$$(i, j, k) \qquad w_i w_j w_k z_L - (-1)^{|i||j|} w_j w_i w_k z_L = [w_i, w_j] w_k z_L.$$

We permute the $i, j, k$ cyclically to obtain the equations

$$\begin{aligned} &(j, k, i) \qquad w_j w_k w_i z_L - (-1)^{|j||k|} w_k w_j w_i z_L = [w_j, w_k] w_i z_L, \\ &(k, i, j) \qquad w_k w_i w_j z_L - (-1)^{|k||i|} w_i w_k w_j z_L = [w_k, w_i] w_j z_L. \end{aligned}$$

By inspection of the ordering of cases in (4.9), we see that equations $(j, k, i)$ and $(k, i, j)$ are true under the inductive assumption. We now show that right-hand side of each of the cyclic permutations above can be expanded in the form,

$$[w_p, w_q] w_r z_L = w_r w_p w_q z_L - (-1)^{|p||q|} w_r w_q w_p z_L + [[w_p, w_q], w_r] z_L. \tag{4.12}$$

For arbitrary $p, q, r \in I$ we have,

$$\begin{aligned} [w_p, w_q] w_r z_L &= \left( d_{pq}{}^{lo} w_l w_o + c_{pq}^k w_k \right) w_r z_L \\ &= d_{pq}{}^{lo} \left( w_r w_l w_o + [w_l, w_r] w_o + w_l [w_o, w_r] \right) z_L + c_{pq}^k \left( w_r w_k + [w_k, w_r] \right) z_L \\ &= w_r \left( d_{pq}{}^{lo} w_l w_o + c_{pq}{}^k w_k \right) z_L + \left( d_{pq}{}^{lo} (w_l [w_o, w_r] + [w_l, w_r] w_o) + c_{pq}^k [w_k, w_r] \right) z_L \\ &= w_r [w_p, w_q] z_L + [[w_p, w_q], w_r] z_L \\ &= w_r w_p w_q z_L - (-1)^{|p||q|} w_r w_q w_p z_L + [[w_p, w_q], w_r] z_L. \end{aligned}$$

Finally, evaluating $(-1)^{|i||k|}(i, j, k) + (-1)^{|j||i|}(j, k, i) + (-1)^{|k||j|}(k, i, j)$, applying (4.12) to the right-hand side of each, we obtain

$$\begin{aligned} \mathbb{A}(w_i, w_j, w_k) z_L = \mathbb{A}&(w_i, w_j, w_k) z_L \\ &+ \left( (-1)^{|i||k|} [[w_i, w_j], w_k] + (-1)^{|j||i|} [[w_j, w_k], w_i] + (-1)^{|k||j|} [[w_k, w_i], w_j] \right) z_L \end{aligned}$$

where $\mathbb{A}(w_i, w_j, w_k)$ is the graded antisymmetric tensor. As result of the graded Jacobi identity this is a trivial identity; hence $(i, j, k)$ is true. $\square$

Note, in particular, that any index ordering such that even elements precede the odd is sufficient for the ordered monomials to form a basis for $U$.

## 4.3 Kac Modules and Atypicality Conditions for Type I′ quadratic Lie superalgebras

In this section we introduce Kac modules for type I′ quadratic Lie superalgebras in a manner completely analogous to that of Lie superalgebras (see section 2.2.4). By definition the odd part of type I′ quadratic superalgebras has the form

$$L_{\overline{1}} = L_- + L_+ \cong V_0(\gamma) + V_0(\gamma^*)$$

for some complex irreducible $L_0$-module with highest weight $\gamma$. The condition $\{L_\pm, L_\pm\} = 0$ is consistent with the $\mathbb{Z}$-gradation

$$L \cong L_0 + L_+ + L_-,$$

where

$$L_+ \cong V_0(\gamma) \qquad L_- \cong V_0(\gamma^*) \qquad L_0 \cong L_{\overline{0}}.$$

Define $U_0 = U(L_0)$, $U_\pm = U(L_\pm)$ as subalgebras of $U = U(L)$. With respect to a basis $\{\overline{Q}^1, \overline{Q}^2, \cdots, \overline{Q}^d,\}$ for $L_+$, and a dual basis $\{Q_1, Q_2, \cdots, Q_d,\}$ for $L_-$, we have

$$\begin{aligned} U_+ &\cong \left\langle \overline{Q}^{i_1}\overline{Q}^{i_2}\cdots\overline{Q}^{i_k},\, i_1 < i_2 < \cdots < i_k,\, 1 \le k \le d \right\rangle, \quad \text{and} \\ U_- &\cong \left\langle Q_{i_1}Q_{i_2}\cdots Q_{i_k},\, i_1 < i_2 < \cdots < i_k,\, 1 \le k \le d \right\rangle, \end{aligned} \tag{4.13}$$

with each of dimension $2^d$ if $\dim(V_0(\lambda)) = d$. The negative and positive odd root spaces $\Delta_{\overline{1}}^-$ and $\Delta_{\overline{1}}^+$ are the weight spaces of $L_-$ and $L_+$ respectively. In this basis the canonical elements (2.47) are

$$T_- = Q_1Q_2\cdots Q_d, \qquad T_+ = \overline{Q}^1\overline{Q}^2\cdots\overline{Q}^d. \tag{4.14}$$

Given the ordering restrictions inherent in the PBW basis theorem for quadratic superalgebras (Theorem 4.2.1), $U$ admits the factorisations

$$U \cong U_0U_-U_+ \cong U_0U_+U_-,$$

with even elements preceding the odd, but not necessarily the factorisation $U = U_-U_0U_+$ or $U = U_+U_0U_-$. Let $V_0(\lambda)$ be a finite-dimensional $L_0$-module with (dominant integral) highest weight $\lambda$. The induced module (2.45), satisfying $L_+V_0(\lambda) = 0$, takes the form

$$\begin{aligned} \overline{V}(\lambda) &= U(L) \otimes_{U(L_0 \oplus L_+)} V_0(\lambda) \\ &= U_0U_-U_+ \otimes_{U(L_0 \oplus L_+)} V_0(\lambda), \end{aligned}$$

where $\otimes_{U(L_0 \oplus L_+)}$ denotes factorisation of the tensor product space by the linear span of elements of the form $gh \otimes v - g \otimes hv$, $g \in U(L)$, $h \in U(L_0 \oplus L_+)$.

**Definition 4.3.1 (Kac modules for quadratic superalgebras)**
*Let $K(\lambda)$ be the maximal invariant submodule of $\overline{V}(\lambda)$. The Kac module $V(\lambda)$ is the irreducible factor module defined as $V(\lambda) := \overline{V}(\lambda)/K(\lambda)$.*

It follows from the definition of $\otimes_{U(L_0\oplus L_+)}$ that

$$\begin{aligned}\overline{V}(\lambda) &= U_0U_-U_+ \otimes_{U(L_0\oplus L_+)} V_0(\lambda)\\ &= U_0U_- \otimes_{U(L_0\oplus L_+)} V_0(\lambda).\end{aligned}$$

Now an arbitrary monomial belonging to $U_0U_-$ has the form

$$x_{i_1}x_{i_2}...x_{i_k}y_{i_{k+1}}y_{i_{k+2}}...y_{i_{k+l}}, \tag{4.15}$$

for which the right-most even generator $x_{i_k}$ can be shifted to the right through each of the odd generators and ultimately through $\otimes_{U(L_0\oplus L_+)}$ resulting, due to the commutation relations, in a sum of terms of the form

$$x_{i'_1}x_{i'_2}...x_{i'_{k-1}}y_{i'_k}y_{i'_{k+2}}...y_{i'_{k+l-1}}.$$

Proceeding similarly for the remaining even terms, the monomial (4.15) is equivalent, modulo $\otimes_{U(L_0\oplus L_+)}$, to a sum of monomials belonging to $U_-$. Thus we have

$$\overline{V}(\lambda) = U_- \otimes_{U(L_0\oplus L_+)} V_0(\lambda).$$

This result allows us to proceed with the following modified construction due to Gould [37], see (2.50).

**Theorem 4.3.2 (Modified Kac module construction by levels ([46] Theorem 3.5))** *Let $V_0(\lambda-2\rho_1)$ be a finite-dimensional $L_0$-module, regarded as a $U(L_0\oplus L_-)$-module by declaring $L_-V_0(\lambda-2\rho_1)=0$. Then the Kac module $V(\lambda)$ is isomorphic to the induced module $UT_+\otimes_{U(L_0\oplus L_-)} V_0(\lambda-2\rho_1)$, that is, we have*

$$V(\lambda) \cong U_-\cdot T_+ \otimes_{U(L_0\oplus L_-)} V_0(\lambda-2\rho_1), \quad or$$
$$V(\lambda) \cong \langle Q_{i_1}Q_{i_2}\cdots Q_{i_k}T_+\otimes v\rangle,\ i_1<i_2<\cdots<i_k,\ 0\le k\le n, v\in V_0(\lambda-2\rho_1). \tag{4.16}$$

**Proof:** See §2.2.4 and references therein.

**Remarks.** Each of the above subspaces of $V(\lambda)$ for $k$ fixed is referred to as the subspace at *level* $k$, $0\le k\le n$. By analogy with the Lie superalgebra case, the module $V(\lambda)$ is said to be *typical* if the level $k=n$ subspace is nontrivial, $\langle T_-T_+\otimes_{U(L_0\oplus L_-)} v, v\in V_0(\lambda-2\rho_1)\rangle \neq \emptyset$; otherwise $V(\lambda)$ is said to be *atypical.*

## 4.4 Quadratic superalgebras as Quadratic Algebras

Let $X$ be a finite-dimensional vector space of dimension $n$ and let $T(X)$ be the tensor algebra generated by $X$. We fix a set of *non-homogeneous quadratic relations* $I\subset(X\otimes X)\oplus X\oplus\mathbb{C}$ to which there corresponds a set of *homogeneous relations* $I_2\subset T\otimes T$ which are the projection of $I$ onto $X\otimes X$. We denote by $(I)$ and $(I_2)$ the ideal generated in $T(X)$ by $I$ and $I_2$ respectively.

**Definition 4.4.1 (Quadratic Algebra)** *Let $X$ and $I$ be defined as above. The algebras*

$$U = T(X)/(I) \qquad \text{and} \qquad A = T(X)/(I_2). \tag{4.17}$$

*are called the* ***inhomogeneous quadratic algebra*** *and the* ***homogenous quadratic algebra*** *respectively, generated by $X$ and $I$.*

Note that the direct sum decomposition $I \subset X \otimes X + X + \mathbb{C}$ enables maps $\alpha : I_2 \to X$ and $\beta : I_2 \to \mathbb{C}$ to be defined such that

$$I = \{x - \alpha(x) - \beta(x) | x \in I_2\}.$$

Within the class of quadratic algebras there exists an important subclass possessing certain cohomological properties which permit a generalisation of the classical PBW theorem. This class contains only homogenous quadratic algebras and is defined by the notion of Koszulness for which there are many equivalent definitions. We give the following definition in terms of the distributivity of certain vector subspaces in the tensor algebra.

**Definition 4.4.2 (Koszul Algebra([61], chapter 2, theorem 4.1))**
*A homogeneous quadratic algebra $A = T(X)/(I_2)$ is Koszul iff for all $n \geq 0$ the collection of subspaces*

$$X^{\otimes i-1} \otimes I_2 \otimes X^{\otimes n-i-1} \subset X^{\otimes n}, \quad i = 1, ..., n-1$$

*is distributive* [1].

As a final preliminary to the generalised PBW theorem we recall the following. Associated with the tensor algebra is the filtration defined by $T_n = \sum_{k=0}^{n} T^k$ in such a way that $\mathbb{C} \cong T_0 \subset T_1 \subset T_2 \subset \cdots$, that is, $T_n \subset T_{n+1}$. $U$ inherits this filtration in the natural way so that we have $\mathbb{C} \cong U_0 \subset U_1 \subset U_2 \subset \cdots$ and we define the *associated graded algebra* as the direct sum

$$grU \equiv \bigoplus_{n}^{\infty} U_n/U_{n-1}. \tag{4.18}$$

Let us compare the construction of $A$ with that of $grU$. $A$, the homogeneous version of $U$, is generated by first homogenising, that is truncating, each term in the generating relations and then factoring the tensor algebra by the resulting ideal. The construction of $grU$, on the other hand, is obtained by initially retaining the full set of non-homogeneous relations and instead truncating the terms appearing in the corresponding ideal. The generalisation of the PBW theorem is a statement of the conditions under which these two graded algebras coincide.

**Theorem 4.4.3 (Generalised PBW theorem([61], chapter 5, theorem 2.1))**
*When $A$ is Koszul, and the following conditions are satisfied:*

$$\begin{array}{ll} (J1) & (\alpha \otimes \mathsf{id} - \mathsf{id} \otimes \alpha)|_{(I_2 \otimes L) \cap (L \otimes I_2)} \subset I_2; \\ (J2) & \alpha \circ (\alpha \otimes \mathsf{id} - \mathsf{id} \otimes \alpha)|_{(I_2 \otimes L) \cap (L \otimes I_2)} = -(\beta \otimes \mathsf{id} - \mathsf{id} \otimes \beta)|_{(I_2 \otimes L) \cap (L \otimes I_2)} \\ (J3) & \beta \circ (\alpha \otimes \mathsf{id} - \mathsf{id} \otimes \alpha)|_{(I_2 \otimes L) \cap (L \otimes I_2)} = 0 \end{array} \tag{4.19}$$

[1] Given a vector space $W$, a collection of subspaces $W_1, W_2, ..., W_N \subset W$ is *distributive* if it generates, with respect to the operations sum and intersection, a lattice of subspaces satisfying the distributivity identity, $(X + Y) \cap Z = X \cap Z + Y \cap Z$, for any triple $X, Y, X$ of its elements.

*we have the isomorphism*

$$gr(U) \cong A.$$

The relations (J1)-(J3) are called the *generalised Jacobi identities*. Proving Koszulness is in general a difficult undertaking and one may select from a variety of methods (for example [61], chapter 2). Here instead we investigate when the homogeneous algebra $A$ of theorem 4.4.3 satisfies the stronger condition of being a *PBW algebra*, that is, admitting an ordered basis of monomials in the generators $x_i \in X$ (see definition 4.4.4). A theorem due to Priddy ([62], theorem 5.3) states that every homogeneous PBW algebra is Koszul.

Before giving a formal definition of a PBW algebra we must make precise the notion of an ordered set of monomials with respect to a set of quadratic relations. Following [62] let $x_i$, $i \in S_1 := \{1, .., n\}$ be a basis for $X$. $S_1$ defines an ordering on $X$ such $x_i < x_j$ when $i < j$. $S_1$ also defines a lexicographic ordering on monomials belonging to $T(X)$. $I_2$ comprises expressions of the form $\sum c^{kl} x_k x_l$. Each such expression contains a unique *leading monomial* which is the highest quadratic term, with respect to the lexicographic ordering, in the sum. We define

$$S_2 := \{(i, j) | i, j \in S_1,\ x_i x_j \text{ is not a leading monomial}\}.$$

We also define

$$S_n := \{(i_1, i_2, ..., i_n) | (i_j, i_{j+1}) \in S_2, j = 1, 2, ..., n-1\}.$$

Consider now the grading of $A$ inherited from $T(X)$. Since $I_2$ is by construction homogeneous, each graded subspace $A_n$ contains only homogeneous elements of degree $n$. It follows that

$$\begin{aligned}
&A_0 \cong \mathbb{C},\\
&A_1 \cong X \text{ has a basis } \{x_i | x_i \in X,\ i \in S_1\},\\
&A_2 \text{ has a basis } \{x_i x_j | x_i, x_j \in X,\ (i, j) \in S_2\}.
\end{aligned}$$

**Definition 4.4.4 (PBW Algebra)** *The homogenous quadratic algebra $A$ is a* PBW-algebra *if the monomials $\{x_{i_1} x_{i_2} ... x_{i_n},\ (i_1, i_2, ..., i_n) \in S_n\}$ are a basis for $A$. In this case the monomials are called a* PBW-basis *of $A$.*

Note that these monomials always span $A$, thus the task of establishing the PBW property is to determine their linear independence. In order to proceed with this task we define the mapping $\pi : X \otimes X \to X \otimes X$

$$\pi(x_i x_j) = \begin{cases} x_i x_j & (i, j) \in S_2 \\ \sum_{\substack{(k,l) < (i,j) \\ (k,l) \in S_2}} c_{ij}^{kl} x_k x_l & (i, j) \notin S_2 \end{cases}$$

which extends linearly to $X \otimes X$ and essentially replaces leading monomials with a unique sum of non-leading terms. The coefficients $c_{ij}^{kl}$ are determined by $I_2$; their existence is guaranteed on general grounds via construction of the corresponding Gröbner basis (see Lemma 1.1 [61]). Finally we define

$$\pi^{12} = \pi \otimes I : T_3 \to T_3. \qquad \pi^{23} = I \otimes \pi : T_3 \to T_3.$$

**Lemma 4.4.5 (Thm 2.1 p.82 [61] - Diamond Lemma)**
*A is a PBW-algebra iff the cubic monomials ($x_ix_jx_k$, $(i,j,k) \in S_3$) are linearly independent in $A_3$. Equivalently A is a PBW-algebra iff the following equation holds:*

$$\cdots \pi^{12}\pi^{23}\pi^{12}\pi^{23}\pi^{12} = \cdots \pi^{23}\pi^{12}\pi^{23}\pi^{12}\pi^{23}. \tag{4.20}$$

**Remarks.** The infinite composition is well defined since $\pi$ decreases the order. To establish (4.20) we need only consider basis elements $x_ix_jx_k \in T_3$ such that both $(i,j),(j,k) \notin S_2$. For if one of these belonged to $S_2$ then one of either $\pi^{12}$ or $\pi^{23}$ will act trivially on the starting term $x_ix_jx_k$ and (4.20) follows immediately. The PBW property depends on the choice of ordering given to the generators of $X$. That is, a fixed homogeneous quadratic algebra may be a PBW algebra given one ordering but not for another.

**Lemma 4.4.6 (Ordering conditions)** *A homogeneous quadratic Lie superalgebra is a PBW-algebra under any index ordering such that only those ${d_{pq}}^{rs}$ are nonvanishing for which k, l precedes p, q.*

**Proof.** The homogeneous relations, obtained by truncation of (4.3), are

$$I_2 = \{x_ix_j - x_jx_i, \quad x_iy_p - y_px_i, \quad y_py_q + y_qy_p - {d_{pq}}^{rs}x_rx_s\}. \tag{4.21}$$

Of these relations, the only ones which differ from the ordinary Lie superalgebra case are

$$y_py_q + y_qy_p - {d_{pq}}^{rs}x_rx_s.$$

The index condition $k$, $l$ precedes $p$, $q$ guarantees that the leading monomial of these relations will always be one of $y_py_q$ or $y_qy_p$ from which it follows that

$$S_2 = \{(i,j)|i \leq j\} \cup \{(i,p)\} \cup \{(p,q)|p < q\} \tag{4.22}$$

where $i,j$ range over all even indices and $p,q$ over all odd indices. Employing Lemma 4.4.5 we need only consider the following elements of $T_3$: $y_ry_qy_p$, $y_ry_qx_i$, $y_rx_jx_i$ and $x_kx_jx_i$ for $p < q < r$ and $i < j < k$. Containing at most one odd generator, the cases $y_rx_jx_i$ and $x_kx_jx_i$ (schematically $\overline{100}$ and $\overline{000}$) satisfy (4.20) as a direct result of the commutativity of even-even and even-odd pairs of generators. Employing the map

$$\pi(y_py_q) = \begin{cases} y_py_q & (p < q) \\ -y_qy_p + {d_{pq}}^{rs}x_rx_s & (p \geq q) \end{cases} \tag{4.23}$$

we examine the remaining two cases in detail. In what follows we evaluate separately the left- and right-hand side of (4.20), applying the operators $\pi^{12}$ and $\pi^{23}$ successively until each side is completely ordered and $\pi$ acts trivially. The notation $\cdots \longrightarrow$ denotes repeated application of these operators in cases where they act simply to reorder commutative pairs. The notation $\{\ \}_{(ordered)}$ is used for expressions containing a summation over even elements; it indicates that the generators comprising each term of the sum are ordered appropriately for the corresponding values of the summation indices.

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$$
\begin{array}{lll}
LHS: & y_r y_q x_i & \overset{\pi^{12}}{\to} (-y_q y_r + d_{rq}{}^{st} x_s x_t) x_i \cdots \longrightarrow - x_i y_q y_r + d_{rq}{}^{st}\{x_s x_t x_i\}_{(ordered)} \\
RHS: & y_r y_q x_i & \overset{\pi^{23}}{\to} y_r x_i y_q \overset{\pi^{12}}{\to} x_i y_r y_q \overset{\pi^{23}}{\to} x_i(-y_q y_r + d_{rq}{}^{st} x_s x_t) \\
 & & \cdots \longrightarrow - x_i y_q y_r + d_{rq}{}^{st}\{x_i x_s x_t\}_{(ordered)}
\end{array}
$$

$\overline{\mathbf{111}}$

$$
\begin{array}{lll}
LHS: & y_r y_q y_p & \overset{\pi^{12}}{\to} (-y_q y_r + d_{rq}{}^{st} x_s x_t) y_p \\
 & & \overset{\pi^{23}}{\to} -y_q(-y_p y_r + d_{rp}{}^{st} x_s x_t) + d_{rq}{}^{st} x_s x_t y_p \\
 & & \overset{\pi^{12}}{\to} (-y_p y_q + d_{qp}{}^{st} x_s x_t) y_r - d_{rp}{}^{st} x_s y_q x_t + d_{rq}{}^{st} x_s x_t y_p \\
 & & \overset{\pi^{23}}{\to} -y_p y_q y_r + d_{qp}{}^{st} x_s x_t y_r - d_{rp}^{st} x_s x_t y_q + d_{rq}{}^{st} x_s x_t y_p
\end{array}
$$

$$
\begin{array}{lll}
RHS: & y_r y_q y_p & \overset{\pi^{23}}{\to} y_r(-y_p y_q + d_{qp}{}^{st} x_s x_t) \\
 & & \overset{\pi^{12}}{\to} -(-y_p y_r + d_{rp}{}^{st} x_s x_t) y_q + d_{qp}{}^{st} x_s y_r x_t \\
 & & \overset{\pi^{23}}{\to} y_p(-y_q y_r + d_{rq}{}^{st} x_s x_t) - d_{rp}{}^{st} x_s x_t y_q + d_{qp}{}^{st} x_s x_t y_r \\
 & & \overset{\pi^{12}}{\to} -y_p y_q y_r + d_{rq}{}^{st} x_s y_p x_t - d_{rp}{}^{st} x_s x_t y_q + d_{qp}{}^{st} x_s x_t y_r \\
 & & \overset{\pi^{23}}{\to} -y_p y_q y_r + d_{rq}{}^{st} x_s y x_t y_p - d_{rp}{}^{st} x_s x_t y_q + d_{qp}^{st} x_s x_t y_r
\end{array}
$$

In both cases we have $LHS = RHS$ thus satisfying (4.20). $\square$

**Lemma 4.4.7 (Jacobi identities for Quadratic Lie superalgebras)**
*The generalised Jacobi identities* (4.19) *are equivalent to* (4.5).

**Proof.** We begin by determining a basis for $(I_2 \otimes L) \cap (L \otimes I_2)$. In the standard Lie algebra case $I_2 \otimes L$ and $L \otimes I_2$ consist of elements of the form $(x_i x_j - x_j x_i) x_k$ and $x_i(x_j x_k - x_k x_j)$ respectively. It is easily seen that the intersection is spanned by cyclic sums of each of these elements which are simply the rank three exterior product, viz

$$
\begin{aligned}
\mathbb{A}(x_i, x_j, x_k) &= (x_i x_j - x_j x_i) x_k + (x_j x_k - x_k x_j) x_i + (x_k x_i - x_i x_k) x_j \\
&= x_i(x_j x_k - x_k x_j) + x_j(x_k x_i - x_i x_k) + x_k(x_i x_j - x_j x_i).
\end{aligned}
$$

Applying (J1) on an arbitrary basis element gives

$$
\begin{aligned}
(\alpha \otimes \mathsf{id} - \mathsf{id} \otimes \alpha)\mathbb{A}(x_i, x_j, x_k) &= c_{ij}{}^l x_l x_k + c_{jk}{}^l x_l x_i + c_{ki}{}^l x_l x_j - c_{ij}{}^l x_k x_l - c_{jk}{}^l x_i x_l - c_{ki}{}^l x_j x_l \\
&= c_{ij}{}^l (x_l x_k - x_k x_l) + c_{jk}{}^l (x_l x_i - x_i x_l) + c_{ki}{}^l (x_l x_j - x_j x_l)
\end{aligned}
$$

which clearly belongs to $I_2$ and imposes no new conditions. Applying (J2) amounts to acting with $\alpha$ on the above yielding,

$$
c_{ij}^l c_{lk}{}^o + c_{jk}{}^l c_{li}{}^o + c_{ki}{}^l c_{lj}{}^o = 0,
$$

which is none other that the ordinary Jacobi identity. For Lie superalgebras the analysis follows similarly, where suitably graded cyclic sums provide a basis for intersection space and

(J2) leads again to the ordinary graded Jacobi identities. For quadratic Lie superalgebras a suitable spanning set is obtained in a similar manner although the satisfaction of (J1) is no longer trivial due to the quadratic expressions $y_p y_q + y_q y_p - d_{pq}{}^{lo} x_l x_o$. We proceed by considering (J1)-(J3) for each of the possible $\mathbb{Z}_2$-graded triples of basis elements, $\overline{111}$, $\overline{110}$, $\overline{100}$ and $\overline{000}$.

Both the $\overline{000}$ and $\overline{001}$ cases are computed without involvement of the additional quadratic term. As a result these simply generate the corresponding Jacobi identities for ordinary Lie superalgebras which are lines one and two of (4.7) respectively. The $\overline{110}$ case leads to cyclic sums of the form

$$\begin{aligned} z_{pqi} &= (y_p y_q + y_q y_p - d_{pq}{}^{k\ell} x_k x_\ell) x_i + (x_i y_q - y_q x_i) y_p + (x_i y_p - y_p x_i) y_q - d_{pq}{}^{k\ell}(x_i x_k - x_k x_i) x_\ell \\ &= x_i (y_p y_q + y_q y_p - d_{pq}{}^{k\ell} x_k x_\ell) + y_p (y_q x_i - x_i y_q) + y_q (y_p x_i - x_i y_p) + d_{pq}{}^{k\ell} x_k (x_i x_\ell - x_\ell x_i) \end{aligned}$$

which are a spanning set for $I_2 \otimes L$ and $L \otimes I_2$. For (J1) we consider

$$\begin{aligned} (\alpha \otimes \mathsf{id} - \mathsf{id} \otimes \alpha)(z_{pqi}) = & \, b_{pq}{}^k x_k x_i + \bar{c}_{iq}{}^r y_r y_p + \bar{c}_{ip}{}^r y_r y_q - d_{pq}{}^{kl} c_{ik}{}^n x_n x_l \\ & - b_{pq}{}^k x_i x_k + \bar{c}_{iq}{}^r y_p y_r + y_q \bar{c}_{ip}{}^r y_r - d_{pq}{}^{kl} c_{il}{}^n x_k x_n, \end{aligned}$$

which in order to expose the interesection with $I_2$ we re-write as:

$$\begin{aligned} (\alpha \otimes \mathsf{id} - \mathsf{id} \otimes \alpha)(z_{pqi}) = & \, b_{pq}{}^k (x_k x_i - x_i x_k) + \bar{c}_{iq}{}^r (y_r y_p + y_p y_r - d_{rp}{}^{lo} x_l x_o) \\ & + \bar{c}_{ip}{}^r (y_r y_q + y_q y_r - d_{rq}{}^{lo} x_l x_o) \\ & + (d_{rp}{}^{lo} \bar{c}_{iq}{}^r x_l x_o + d_{rq}{}^{lo} \bar{c}_{ip}{}^r x_l x_o - d_{pq}{}^{kl} c_{ik}{}^n x_n x_l - d_{pq}{}^{kl} c_{il}{}^n x_k x_n) \end{aligned}$$

where new terms $d_{rp}{}^{lo} \bar{c}_{iq}{}^r x_l x_o$ and $d_{rq}{}^{lo} \bar{c}_{ip}{}^r x_l x_o$ have been added and subtracted. The fulfillment of (J1) demands that the last term vanish; this is precisely the third equation of (4.7). Proceeding with (J2), we act with $\alpha$ on the first three terms above which yields the fourth equation of (4.7).

Finally the $\overline{111}$ case leads to the fifth and sixth equations of (4.7) which are derived from (J1) and (J2) respectively. In this case vector space $I_2 \otimes L \cap L \otimes I_2$ is spanned by the elements

$$\begin{aligned} z_{pqr} &= -d_{qr}{}^{lo}(y_p x_l - x_l y_p) x_o - d_{rp}{}^{lo}(y_q x_l - x_l y_q) x_o - d_{pq}{}^{lo}(y_r x_l - x_l y_r) x_o \\ &\quad + (y_p y_q + y_q y_p - d_{pq}{}^{lo} x_l x_o) y_r + (y_q y_r + y_r y_q - d_{qr}{}^{lo} x_l x_o) y_p + (y_r y_p + y_p y_r - d_{rp}{}^{lo} x_l x_o) y_q \\ &= -d_{qr}{}^{lo} x_l (x_o y_p - y_p x_o) - d_{rp}{}^{lo} x_l (x_o y_q - y_q x_o) - d_{pq}{}^{lo} x_l (x_o y_r - y_r x_o) \\ &\quad + y_r (y_p y_q + y_q y_p - d_{pq}{}^{lo} x_l x_o) + y_p (y_q y_r + y_r y_q - d_{qr}{}^{lo} x_l x_o) + y_q (y_r y_p + y_p y_r - d_{rp}{}^{lo} x_l x_o), \end{aligned}$$

and the results follow directly following the procedure above. □

**Lemma 4.4.8 (PBW Theorem for Quadratic Lie Superalgebras.)**
*Under the ordering conditions of Lemma 4.4.6 a quadratic Lie superalgebra has a PBW basis of ordered monomials.*

**Alternative Proof for Theorem 4.2.1.** Lemma 4.4.6 establishes the ordering conditions and the PBW property of the corresponding homogeneous algebra. Lemma 4.4.7 proves the equivalence of the generalised Jacobi identities. It follows that the required monomials are inherited from the homogeneous algebra due to Theorem 4.4.3. □

Chapter 5

# THE QUADRATIC SUPERALGEBRA $gl_2(n/1)$

This chapter begins with the definition and analysis of a class of type I′ quadratic superalgebras which we denote $gl_2(n/1)$. This class, originally introduced without a PBW basis theorem in [45], may be viewed as a quadratic deformation of the class $sl(n/1)$ of ordinary Lie superalgebras. Employing the PBW theorem and associated lemmas established in the previous chapter we extend the results of [45] to define a one-parameter family of PBW algebras up to an overall central charge. The present analysis of $gl_2(n/1)$ includes a derivation and clarification of admissible structure constants originally given in [45], the degeneracy of the $n = 2$ case to $sl(2/1)$ and an initial investigation into atypicality conditions for certain classes of representations. In §5.4 broader constructions are considered; these include an enlargement of the class $gl_2(n/1)$ via the imposition of additional structural relations and a mild generalisation of the action of the even subalgebra on the odd generators.

Results in this chapter have been published in the jointly authored papers [46, 82]. Unless otherwise stated, the results presented in this chapter, are in the largest proportion, due to the candidate; in particular Lemmas 5.1.1 and 5.1.2 which concern the contraction limit of $gl_2(n/1)$ and the degeneracy of the $n = 2$ cases respectively, §5.3 and Lemma 5.3.1 which analyse zero-step atypicals, and Lemma 5.4.2 which proves the failure of the PBW property in the event of imposing additional covariant structural relations. §5.2 concerning single-step atypicality conditions and Lemmas 5.4.1 and 5.4.3 are due largely to Jarvis [46, 82].

## 5.1 Definitions and the Derivation of Structure Constants

We define $gl_2(n/1)$ as a quadratic deformation of the ordinary Lie superalgebra $sl(n/1)$. As such the even and odd parts are

$$L_{\overline{0}} \cong gl(n) \cong sl(n) + gl(1) \tag{5.1}$$

$$L_{\overline{1}} = L_{-1} + L_{+1} \cong \{1\} + \{\overline{1}\} \tag{5.2}$$

where $\{1\}$ and $\{\overline{1}\}$ are the fundamental and fundamental contragredient $L_{\overline{0}}$-representations respectively. Take as a basis for $gl(n)$ the Gel'fand generators $E^a{}_b$ $a, b = 1, ..., n$ satisfying (2.19). The odd generators are $\overline{Q}^a$ and $Q_a$ for $L_{+1}$ and $L_{-1}$ respectively. Under the adjoint action of $gl(n)$ these transform as vector operators satisfying

$$[E^a{}_b, \overline{Q}^c] = \delta^c{}_b \overline{Q}^a \qquad [E^a{}_b, Q_c] = -\delta^a{}_c Q_b. \tag{5.3}$$

Our aim is to determine the most general solution for the structure constants of the quadratic anticommutator

$$\{\overline{Q}^a, Q_b\} \subset L_{\overline{0}} \otimes L_{\overline{0}} + L_{\overline{0}} + \mathbb{C}.$$

By theorem 4.4.3, $gl_2(n/1)$ is a PBW algebra when the generalised Jacobi identities (4.19) are satisfied. Theorem 4.4.7 tells that these are equivalent to the ordinary Jacobi identities, which when evaluated in terms of homogeneous graded basis elements leads to four distinct cases $\overline{000}$, $\overline{001}$, $\overline{011}$ and $\overline{111}$. The first two cases are naturally satisfied by $gl_2(n/1)$; $\overline{000}$ states that $L_{\overline{0}}$ is a Lie algebra and $\overline{001}$ expresses the covariance of $L_{\overline{1}}$ under the adjoint action of $L_{\overline{0}}$.

The $\overline{011}$-case takes the form

$$[E^a{}_b, \{\overline{Q}^c, Q_b\}] = \{[E^a{}_b, \overline{Q}^c], Q_b\} + \{\overline{Q}^c, [E^a{}_b, Q_b]\} \tag{5.4}$$

expressing the requirement that $\{L_{\overline{1}}, L_{\overline{1}}\}$ also transforms covariantly under the adjoint action of $L_{\overline{0}}$. More specifically, to compute admissible structure constants we require the quadratic and linear combinations of (adjoint) $L_{\overline{0}}$-operators belonging the right-hand-side of the anticommutator transform as irreducible submodules of $L_{\overline{0}} \otimes L_{\overline{0}}$ and $L_{\overline{0}}$ which are common to those of the *symmetric* product of $L_{\overline{1}}$ with itself. The branching multiplicities of the common irreducible submodules will determine the number and type of generalised structure constants.

It is well-known that the tensor product of the fundamental representation with its contragredient decomposes into the sum of the (traceless) $(n^2 - 1)$-dimensional part $\mathrm{ad}_{L_{\overline{0}}}$, and the 1-dimensional linear Casimir invariant; this can easily be computed using the method of Young diagrams as presented in section 2.1.4. Following [45], the symmetric projection of $\mathrm{ad}_{L_{\overline{0}}} \otimes \mathrm{ad}_{L_{\overline{0}}}$ contains both the adjoint representation and the linear Casimir each with a multiplicity of 2; this results in $2 \times 2 = 4$ independent coefficients associated with quadratic terms. The linear terms naturally comprise of one copy of $\mathrm{ad}_{L_{\overline{0}}}$ together with its trace, contributing 2 linear coefficients, and the (scalar) central extension contributes one final coefficient. Thus the general anticommutator contains 7 terms, or 6 arbitrary parameters up to an overall normalisation.

There exists a general method for determining the form of each term at a given monomial degree[1].The terms are always of the form $(E^m)^a{}_b \langle E^\kappa \rangle$ where $E^m$ is the standard matrix

[1] The method employs Joseph's theorem which states that the universal enveloping algebra, viewed as an $L_{\overline{0}}$-module, is isomorphic to the direct sum of the tensor product of every dominant integral $L_{\overline{0}}$-representation with its contragredient. Irreducible submodules, differentiated by degree and symmetry type, belonging to each of these products may then be expressed as polynomials in the adjoint operators (see section 2 [45] and references therein).

power, and $\kappa$ is a partition of $\ell = 0, 1, 2$ written in component form as $(2^{\kappa_2}1^{\kappa_1})$ such that

$$\langle E^\kappa \rangle = \langle E^2 \rangle^{\kappa_2} \langle E \rangle^{\kappa_1}, \tag{5.5}$$

where $\langle E^k \rangle$ denotes the trace of $E^k$, that is, the degree $k$ Casimir operator. Writing each form symbolically as $m(\kappa)$, the required linear and quadratic terms are those combinations for which $m + \ell = 1, 2$ respectively, viz

$$\begin{array}{lll}
\text{Symbol} & \text{Term} & \text{Commutator} \\
\hline
1(0) & E^a{}_b & [E^a{}_b, Q_c] = -\delta^a{}_c Q_b \\
0(1) & \langle E \rangle & [\langle E \rangle, Q_c] = -Q_c \\
 & & \\
2(0) & (E^2)^a{}_b & [(E^2)^a{}_b, Q_c] = -E^a{}_c Q_b - \delta^a{}_c (QE)_b \\
1(1) & E^a{}_b \langle E \rangle & [E^a{}_b \langle E \rangle, Q_c] = -\delta^a{}_c Q_b \langle E \rangle - E^a{}_b Q_c \\
0(2) & \langle E^2 \rangle & [\langle E^2 \rangle, Q_c] = -2(QE)_c + nQ_c \\
0(1^2) & \langle E \rangle^2 & [\langle E \rangle^2, Q_c] = -2Q_c \langle E \rangle + Q_c,
\end{array} \tag{5.6}$$

where $(QE)_j := Q_k E^k{}_j$. Thus, the fulfillment of the $\overline{0}11$ Jacobi identity leads to the general form [2]

$$\{\overline{Q}^a, Q_b\} = (E^2)^a{}_b + \mathsf{a}\langle E^2 \rangle \delta^a{}_b + (\mathsf{b}_1 \langle E \rangle + \mathsf{b}_2) E^a{}_b + (\mathsf{c}_1 \langle E \rangle^2 + \mathsf{c}_2 \langle E \rangle + \mathsf{c}) \delta^a{}_b. \tag{5.7}$$

Finally the $\overline{1}11$-case is imposed to determine constraints on the above coefficients. The identity takes the following two forms:

$$\begin{aligned}
[\{\overline{Q}^a, Q_b\}, Q_c] &= [\overline{Q}^a, \{Q_b, Q_c\}] - [\{\overline{Q}^a, Q_c\}, Q_b] \equiv -[\{\overline{Q}^a, Q_c\}, Q_b] \\
[\{Q_a, \overline{Q}^b\}, \overline{Q}^c] &= [Q_a, \{\overline{Q}^b, \overline{Q}^c\}] - [\{Q_a, \overline{Q}^c\}, \overline{Q}^b] \equiv -[\{Q_a, \overline{Q}^c\}, \overline{Q}^b],
\end{aligned} \tag{5.8}$$

where owing to the type I$'$ condition $\{L_{+1}, L_{+1}\} = \{L_{-1}, L_{-1}\} = 0$ the first and second form may be viewed as an antisymmetry condition under the exchange of the two covariant and contravariant vector indices respectively. Using (5.6) and (5.7) we expand $[\{\overline{Q}^a, Q_b\}, Q_c]$ and collect the various terms to identify the relevant coefficients:

$$\begin{array}{rlrl}
\delta^a{}_c Q_b : & -\mathsf{b}_2 & \delta^a{}_b Q_c : & -\mathsf{c}_2 + n\mathsf{a} + \mathsf{c}_1 \\
E^a{}_c Q_b : & -1 & E^a{}_b Q_c : & -\mathsf{b}_1 \\
\delta^a{}_c (QE)_b : & -1 & \delta^a{}_b (QE)_c : & -2a \\
\delta^a{}_c Q_b \langle E \rangle : & -\mathsf{b}_1 & \delta^a{}_b Q_c \langle E \rangle : & -2\mathsf{c}_1.
\end{array} \tag{5.9}$$

Despite the existence of a common monomial term amongst certain of these expressions, it is easily shown that we may project onto them, thus obtaining independent equations, due to constraints arising from the limited number of free parameters compared with the number of independent monomial terms. This also applies to the smallest case, $n = 2$, where the number of monomial terms in minimal. Hence, there are four antisymmetry conditions and five coefficients. Setting $-\mathsf{b}_2 = \alpha$ yields the constraints

$$\begin{aligned}
\mathsf{a} &= -\frac{1}{2} & \mathsf{b}_1 &= -1 \\
\mathsf{c}_1 &= \frac{1}{2} \quad \text{and} & \mathsf{c}_2 &= \alpha - \frac{1}{2}\mathsf{n} + \frac{1}{2}
\end{aligned} \tag{5.10}$$

[2] Without any loss of generality, the leading coefficient has been set to 1 since the anticommutator may be rescaled by an overall factor $\lambda$ as a result of the basis transformation $Q_a \to \sqrt{\lambda} Q_a$, $\bar{Q}^a \to \sqrt{\lambda} \bar{Q}^a$.

with no restriction on $\mathsf{c}$. This determines a one-parameter family $gl_2(n/1)^{\alpha,\mathsf{c}}$, up to a central charge, of PBW algebras with the anticommutator given by [3],

$$\{\overline{Q}^a, Q_b\} = (E^2)^a{}_b - E^a{}_b(\langle E\rangle - \alpha) - \frac{1}{2}\delta^a{}_b\left(\langle E^2\rangle - \langle E\rangle^2 + (n-1+2\alpha)\langle E\rangle\right) + \delta^a{}_b\mathsf{c}. \quad (5.11)$$

Given that the structure constants for the graded multiplication $[L_{\bar{0}}, L_{\bar{0}}] \subset L_{\bar{0}}$ and $[L_{\bar{0}}, L_{\pm 1}] \subset L_{\pm 1}$ are fixed by the choice of $L_{\bar{0}}$ and $L_{\bar{1}}$, (5.11) represents the most general solution possible for a quadratic anticommutator that fulfills the condition of being a PBW algebra (see Definition 4.4.4). The family of quadratic superalgebras $gl_2(n/1)^{\alpha,\mathsf{c}}$ possesses an ordered PBW basis which, in the absence of any further constraints, comprise the ordered monomials under the ordering condition of Lemma 4.4.6. In particular, any ordering such that the even elements precede the odd is sufficient to meet this condition.

The close relationship between $gl_2(n/1)$ and the Lie superalgebra $sl(n/1)$ is revealed by a simple rescaling of the odd elements (see footnote on the previous page). We transform the odd elements, rescaling each by the factor $\sqrt{\lambda}$, explicitly $Q_a \to \sqrt{\lambda}Q_a$ and $\bar{Q}^a \to \sqrt{\lambda}\bar{Q}^a$, where we set $\lambda \equiv \frac{1}{\alpha}$. Let us denote by $gl_2(n/1)^{\lambda,\mathsf{c}}$ the re-parametrised quadratic superalgebra that results from this basis transformation; the resulting anticommutator is rescaled by a factor of $\lambda$, explicitly

$$\{\overline{Q}^a, Q_b\} = \lambda\big[(E^2)^a{}_b - E^a{}_b\langle E\rangle - \tfrac{1}{2}\delta^a{}_b\big(\langle E^2\rangle - \langle E\rangle^2 + (n-1)\langle E\rangle - 2\mathsf{c}\big)\big] + E^a{}_b - \delta^a{}_b\langle E\rangle.$$

**Lemma 5.1.1 (Contraction limit of $gl_2(n/1)^{\lambda,\mathsf{c}}$)** *The ordinary Lie superalgebra $sl(n/1)$ is obtained from $gl_2(n/1)^{\lambda,\mathsf{c}}$ in the contraction limit $\lambda \to 0$.*

**Lemma 5.1.2 (Degeneracy of the $n = 2$ quadratic case)** *The quadratic Lie superalgebra $gl_2(2/1)$ is degenerate and isomorphic to $sl(2/1) \cong A(1,0)$.*

**Proof.** Evaluation of (5.11) for the $n = 2$ case yields the linear relations:

$$\begin{aligned} \{\overline{Q}^1, Q_1\} &= -E^2{}_2 + \mathsf{c}, \qquad & \{\overline{Q}^1, Q_2\} &= -E^1{}_2, \\ \{\overline{Q}^2, Q_1\} &= E^2{}_1, & \{\overline{Q}^2, Q_2\} &= -E^1{}_1 + \mathsf{c}. \end{aligned}$$

Identifying $\frac{1}{2}(E^1{}_1 - E^2{}_2) \to H$ and $-\frac{1}{2}(E^1{}_1 + E^2{}_2) + c \to Z$ the above relations may be expressed as

$$\begin{aligned} \{\overline{Q}^1, Q_1\} &= Z + H, \qquad & \{\overline{Q}^1, Q_2\} &= -E^1{}_2, \\ \{\overline{Q}^2, Q_1\} &= E^2{}_1, & \{\overline{Q}^2, Q_2\} &= Z - H. \end{aligned}$$

The remaining commutation relations between $H$, $Z$, the two even root vectors $E^1{}_2$, $E^2{}_1$ and the four odd vector operators are easily calculated in the Gel'fand basis generating the standard Cartan-Weyl presentation of $sl(2/1)$. □

[3]This solution is implicit in [45] however the original presentation obscured the free parameter by fixing $\alpha = 0$ due to a certain rescaling of the linear Casimir. The implications of this on the parameter associated with the central charge $\mathsf{c}$ were also overlooked (see Lemma A.1.1). Additional free parameters are obtained in [45], although in this case due to the imposition of additional structural relations; these turn out to remove the PBW property (see Lemma 5.4.2). Broader constructions along the lines are examined in §5.4

## 5.2 Atypicality Conditions

In view of Theorem 2.2.1 for Lie superalgebras and its extension to Theorem 4.3.2 for $gl_2(n/1)$, we have via the modified Kac module a constructive approach to investigate the structure of atypical representations. In practice the problem is still difficult, even for single-step atypical modules, because the action of a given $Q_i \in L_{-1}$ on an arbitrary element $T_+ \otimes v$ for $v \in V_0(\lambda - 2\rho_1)$ projects in general onto any one of the $L_{\bar{0}}$-modules which comprise the Kac module $V(\lambda - \alpha_i)$, $i = 1, ..., n$.

In the case of Lie superalgebras this problem has been resolved, at least for $gl(n/1)$ and certain classes of $gl(m/n)$ representations, by employing the techniques of tensor projection operators derived from polynomial identities [39, 38, 37]. We review and apply this method to $gl_2(n/1)$ restricting the analysis, for simplicity, to single-step modules[4]. The key idea is that single-step modules comprise irreducible $L_{\bar{0}}$-modules belonging to the decomposition of the tensor product $Y \equiv V_0(\varphi) \otimes V_0(\lambda)$ where $V_0(\varphi) \cong L_{-1}$ is the fundamental contragredient representation. In view of the theory of characteristic identities presented in Section 3.2, the matrix array $E \in End(Y)$ of Gel'fand generators satisfies an identity of (maximal) degree $n$ given by

$$P(E) = \prod_{i=1}^{n} (E - a_i) \tag{5.12}$$

where $a_i = \lambda_i + n - 1$ is given by (3.47) for highest weight $\lambda = \sum_i \lambda_i \varepsilon_i$. As per the discussion in §3.2.3 $E$ may satisfy an identity of reduced degree where, in the case that $V(\mu)$ is finite-dimensional, the factor $(E - a_i)$ remains in the product (5.12) only if $\lambda + \varphi_i$ is dominant integral. For infinite-dimensional representations the situation is more complicated due to the fact the $Y$ does not necessarily decompose into the sum of ordinary eigenspaces of the invariant $E$; the resulting minimal identity is given by (3.29). Notwithstanding the difficulties associated with determining the minimal identity, we proceed, following [39] (see also [35]), to define the projection operators for finite-dimensional $V(\lambda)$[5] ,

$$P[s] \equiv \frac{\prod_{s' \neq s} (E - a_{s'})}{\prod_{s' \neq s} (a_s - a_{s'})}. \tag{5.13}$$

These satisfy $P[s]P[t] = \delta_{st} P[t]$, $\mathsf{Id} = \sum_{s=1}^{n} P[s]$ and $Y_s = P[s]Y$ (where $Y_s$ is defined by (3.27)). In view of the characteristic identity 5.12, we have by construction,

$$E^i{}_k P[s]^k{}_j = a_s P[s]^i{}_j. \tag{5.14}$$

The projectors allow each vector operator $Q_i \in L_{-1}$ to be resolved into a sum of so-called shift operators,

$$Q_i = \sum_{s=1}^{n} Q_i[s], \qquad Q_i[s] \equiv Q_k P[s]^k{}_i. \tag{5.15}$$

[4]Higher order $k$-step modules are in principle computable using combinatorial manipulations which generalise the present methods. For example, the second-level atypicality conditions would rely on the shift operators (compare (5.16)) of the form $Q_{ij}[s,t] = Q_i[s]Q_j[t] + Q_i[t]Q_j[s]$; these are the projected parts of $Q_i Q_j \overline{v}$ onto the even submodule $V_0(\lambda - \alpha_s - \alpha_t)$. See §A.3 [46] for further discussion.

[5]A definition of projection operators for infinite-dimensional representations is given in §A.4.

The action of each component $Q_i[s]$ on an arbitrary $\overline{v} \equiv T_+ \otimes v$, for $v \in V_0(\lambda - 2\rho_1)$, *shifts* the vector into the module $V_0(\lambda - \alpha_s)$, viz

$$Q_i[s]\overline{v} = Q_k P[s]^k{}_i \overline{v} \in V_0(\lambda - \alpha_s) \qquad s = 1, ..., n. \tag{5.16}$$

To proceed more concretely, we introduce the following quantities expressed with the aid of the totally antisymmetric Levi-Cevita tensor $\varepsilon_{i_1 i_2 \cdots i_n}$ [38]:

$$\begin{aligned}
\overline{T} &= \frac{1}{n!}\varepsilon_{i_1 i_2 \cdots i_n} \overline{Q}^1 \overline{Q}^2 \cdots \overline{Q}^n, \\
\overline{T}_{i_1 i_2 \cdots i_k} &= \frac{(-1)^{\frac{1}{2}k(k-1)}}{(n-k)!}\varepsilon_{i_1 i_2 \cdots i_k i_{k+1} i_n} \overline{Q}^1 \overline{Q}^2 \cdots \overline{Q}^n.
\end{aligned} \tag{5.17}$$

We note that $\overline{T}$ is identical to $T_+$ and that $\overline{T}_i$ is equivalent, up to an overall normalisation, to $\overline{T}$ with the element $\overline{Q}^i$ removed from the product. These elements satisfy the following relations:

$$\begin{aligned}
& \overline{Q}^i \overline{T} = 0, \\
& \overline{Q}^i \overline{T}_j = \delta^i{}_j \overline{T}, \\
& \overline{Q}^i \overline{T}_{jk} = \delta^i{}_j \overline{T}_k - \delta^i{}_k \overline{T}_j, \\
& \overline{Q}^i \overline{T}_{jk\ell \cdots} = \delta^i{}_j \overline{T}_{k\ell \cdots} - \delta^i{}_k \overline{S}_{j\ell \cdots} + \delta^i{}_\ell \overline{T}_{jk \cdots} + \cdots, \\
& [E^i{}_j, \overline{T}_{i_1 i_2 \cdots i_k}] = \delta^i{}_j \overline{T}_{i_1 i_2 \cdots i_k} - \delta^i{}_{i_1} \overline{T}_{j i_2 \cdots i_k} - \cdots, \\
\text{and} \quad & [\langle E \rangle, \overline{T}_{i_1 i_2 \cdots i_k}] = (n-k) \overline{T}_{i_1 i_2 \cdots i_k}.
\end{aligned} \tag{5.18}$$

With these quantities in hand we continue the evaluation of the single-step atypicality conditions following on from (5.16),

$$\begin{aligned}
Q_i[s]\overline{v} &= Q_k \overline{T} \otimes P[s]^k{}_i v \\
&= \big([Q_k, \overline{T}] - \overline{T} Q_k\big) \otimes P[s]^k{}_i v \\
&= [Q_k, \overline{T}] \otimes P[s]^k{}_i v \in V_0(\lambda - \alpha_s) \qquad s = 1, ..., n.
\end{aligned} \tag{5.19}$$

This indicates that the module $V_0(\lambda - \alpha_s)$ will belong to $V(\lambda)$ whenever $[Q_k, \overline{T}]$ acts non-trivially on the projected component $P[s]^k{}_i v$. In the case of $gl(n/1)$ we have the following result due to [38],

$$[Q_k, \overline{T}] = \overline{T}_l A^l{}_k, \tag{5.20}$$

where $A^l{}_k = (E + Z - n + 1)^l{}_k$. Inserting (5.20) into (5.19) yields,

$$\begin{aligned}
[Q_k, \overline{T}] \otimes P[s]^k{}_i v &= \overline{T}_l (E + Z - n + 1)^l{}_k \otimes P[s]^k{}_i v \\
&= (a_s + z - n + 1) \overline{T}_l \otimes P[s]^l{}_i v \\
&= \langle \lambda + \rho, \alpha_s \rangle \overline{T}_l \otimes P[s]^l{}_i v \in V_0(\lambda - \alpha_s).
\end{aligned} \tag{5.21}$$

where the final line uses $\langle \lambda + \rho, \alpha_s \rangle = a_s + z - n + 1$. Thus, $V_0(\lambda - \alpha_s)$ occurs in $V(\lambda)$ if and only if $\lambda - \alpha_s$ is dominant integral and $\langle \lambda + \rho, \alpha_s \rangle \neq 0$, $s = 1, ..., n$. This result is applied to the $gl(2/1)$ example given at the end of Chapter 2, see (2.61).

Following [46], we adapt the techniques above and apply them to $gl_2(n/1)$ taking the definitions (5.17) and (5.19) as our starting place. The point of differentiation in the quadratic case is the evaluation of ${A^l}_k$ in (5.20). Observe that $\overline{Q}^m\overline{T} = 0$ implies

$$0 = [Q_k, \overline{Q}^m\overline{T}] = \{Q_k, \overline{Q}^m\}\overline{T} - \overline{Q}^m[Q_k, \overline{T}].$$

Thus we have

$$\begin{aligned}\{Q_k, \overline{Q}^m\}\overline{T} = \overline{Q}^m[Q_i, \overline{T}] &= \overline{Q}^m\overline{T}_l{A^l}_k \\ &= {\delta^m}_l\overline{T}{A^l}_k \\ &= \overline{T}{A^m}_k,\end{aligned}$$

hence, using (5.11),

$$\overline{T}{A^i}_j = \left((E^2)^i{}_j - {E^i}_j(\langle E\rangle - \alpha) - \frac{1}{2}{\delta^i}_j\left(\langle E^2\rangle - \langle E\rangle^2 + (n-1+2\alpha)\langle E\rangle + c\right)\right)\overline{T}. \quad (5.22)$$

To obtain ${A^i}_j$ explicitly we need to shuffle $\overline{T}$ to the left-hand side of the bracket. Using ${E^i}_j\overline{T} = \overline{T}({E^i}_j + n{\delta^i}_j)$ and $\overline{T}_k{E^k}_i = \left((n-1)\delta_i{}^k - E_i{}^k\right)\overline{T}_k$ the following result is obtained

$${A^i}_j = (E^2)^i{}_j - (\langle E\rangle + (n-2) - \alpha){E^i}_j - \tfrac{1}{2}{\delta^i}_j\left(\langle E^2\rangle - \langle E\rangle^2 - (n-3+2\alpha)\langle E\rangle\right) + (c-(n-1)){\delta^i}_j.$$

**Lemma 5.2.1 (First-level atypicality conditions for $gl_2(n/1)$ ([46] Theorem 4.3))** *Take $L = gl_2(n/1)$ and let $V_0(\lambda)$ be an arbitrary finite-dimensional $L_0$-module. Let $\varphi_s$ be a weight of $L_{-1}$ viewed as an $L_0$-module. The irreducible Kac module $V(\lambda)$, induced from $V_0(\lambda)$, contains the $L_0$-submodule $V_0(\lambda-\alpha_s)$ iff $\lambda+\varphi_s$ is dominant integral and $a_s(\lambda_1, \lambda_2, ..., \lambda_n) \neq 0$ for a certain polynomial function in the highest weight labels.*

**Proof.** The requirement that $\lambda + \varphi_s$ be dominant integral is evident since the induced $L$-module is necessarily finite-dimensional. We apply the method outlined above as used in Gould [37] and Gould et al. [38] for the Lie superalgebra $gl(n/1)$. Using the definitions and rearrangement identities we substitute (5.22) into (5.20) and obtain the quadratic adaptation of (5.21) which determines the projected spanning states of the level 1 subspace:

$$\begin{aligned}Q_i[s]\overline{v} &= [Q_k, \overline{T}] \otimes P[s]^k{}_i v \\ &= \overline{T}_l{A^l}_k \otimes P[s]^k{}_i v \\ &\equiv a_s(\lambda_1, \lambda_2, ..., \lambda_n)\overline{T}_l \otimes P[s]^l{}_i v \in V_0(\lambda - \alpha_s) \quad (5.23)\end{aligned}$$

where in the last step the property (5.14) reduces ${A^i}_j$ to the polynomial form $a_s(\lambda_1, \lambda_2, ..., \lambda_n)$ by substituting the root $a_s{\delta^i}_j$ for ${E^i}_j$, where for $\lambda' = \lambda - 2\rho_1$ we have $a_s = \lambda'_s + n - s$ (see (3.47)), and the usual Casimir eigenvalues for $\langle E\rangle$ and $\langle E^2\rangle$ evaluated on the $L_0$-module $V_0(\lambda - 2\rho_1)$ carried by the states $v$:

$$\langle E\rangle \mapsto \sum_{i=1}^{n} \lambda'_i \qquad\qquad \langle E^2\rangle \mapsto \sum_{i=1}^{n} \lambda'_i(\lambda'_i + n + 1 - 2i).$$

$\square$

## 5.3 Zero-step Modules

A remarkable feature of quadratic Lie superalgebras is the existence of a class of truncated atypical representations where, in the event that the even subalgebra satisfies a minimal polynomial identity of degree 2, the right-hand-side of the anticommutator (5.11) may in certain cases be identically zero. Representations with this property, denoted as *zero-step* atypicals, comprise a single $L_0$-module, the top module $V_0(\lambda) = \overline{T}v$, $v \in V_0(\lambda - 2\rho_1)$. As an illustration of this feature we examine in this section a restricted class of finite-dimensional $gl(n)$-representations in which the existence of a quadratic identity is guaranteed. In the following chapter we shall continue along these lines, investigating a certain class of infinite-dimensional representations for the case $n = 4$.

We start by considering candidate $L_0$-modules which satisfy a degree 2 polynomial identity. Let $V_0(\varphi)$ be the module corresponding to the fundamental contragredient representation with distinct weights $\varphi_1, \varphi_2, ..., \varphi_n$ and let $V(\lambda)$ be an arbitrary finite-dimensional $L_0$-module. It is well-known[6] that highest weights occurring in decomposition $V_0(\varphi) \otimes V_0(\lambda)$ are all the dominant integral weights of the form $\lambda + \varphi_i$. Taking $V_0(\varphi)$ as the reference representation the corresponding minimal polynomial is given by (3.30), where the degree is equal to the number of distinct $\lambda + \varphi_i$.

In partition notation we have $\varphi_i = (1^{n-i}, 0, 1^{i-1})$ which, when added to $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, amounts to adding 1 to all but the $\lambda_{n+1-i}$ component of $\lambda$. In terms of Young diagrams this is equivalent to adding one box to all but the $(n+1-i)^{\text{th}}$-row of the $\lambda$-partition. For the remainder of this section we fix,

$$\lambda = (k^r, 0^{n-r}) \cong \overbrace{\begin{array}{|c|c|c|c|c|c|c|c|}\hline \; & \; & \; & \; & \; & \; & \; & \; \\ \hline \; & \; & \; & \; & \; & \; & \; & \; \\ \hline \; & \; & \; & \; & \; & \; & \; & \; \\ \hline \; & \; & \; & \; & \; & \; & \; & \; \\ \hline \end{array}}^{k \text{ columns}} \left.\vphantom{\begin{array}{c} \\ \\ \\ \\ \end{array}}\right\} r \text{ rows}, \tag{5.24}$$

for which the Casimir operators, evaluated on $V_0(\lambda)$, take the eigenvalues

$$\langle E \rangle = rk \qquad \langle E^2 \rangle = rk(k + n - r). \tag{5.25}$$

Given the requirement that $\lambda + \varphi_i$ be dominant integral it is evident that $\lambda + \varphi_{n-r+1}$ and $\lambda + \varphi_1$ are the only possible choices. These correspond to the addition of one box to all the rows of $\lambda$ except the $r^{th}-$ and $n^{th}-$rows respectively. The resulting quadratic $gl(n)$-polynomial identity is, using (3.47),

$$E(E - (k + n - r)) = 0. \tag{5.26}$$

This result agrees with the examples given for small values of $r$ in equation (48) Green [39]. Zero-step modules will occur whenever (5.26) coincides with the right-hand side of the anticommutator (5.11). This leads to the following two conditions:

$$\alpha = \langle E \rangle - k - n + r \tag{5.27}$$

$$0 = \langle E^2 \rangle - \langle E \rangle^2 + (n - 1 + 2\alpha)\langle E \rangle - 2\mathsf{c}. \tag{5.28}$$

[6] See for example Gould p.7 [36] or Kostant p.259 [51]

Substituting the first into the second and replacing the Casimir operators with their respective eigenvalues proves the following result.

**Lemma 5.3.1 (Zero-step atypicals for $gl_2(n/1)$ Kac modules $V(k^r, 0^{n-r})$)**
*The $gl_2(n/1)$ Kac module $V(k^r, 0^{n-r})$ for $n = 3, 4, 5, ...$, $k = 1, 2, 3, ...$ and $r = 1, 2, 3, ..., n-1$ is a zero-step atypical when*

$$c = \frac{1}{2}r(r-1)k(k+1), \tag{5.29}$$

*where $c$ is the central extension of the $gl_2(n/1)$ anticommutator given by* (5.11).

The $c = 0$ case restricts $r = 1$ which is the class of symmetric representations characterised by $\lambda = (k, 0, 0, ..)$. The following table is easily computed using (5.25) and (5.27), and gives concrete instances for the restricted case $\alpha = 0$ including all those published in [46].

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 3 | 4 | 5 | 5 | 6 | 7 | 7 | 7 | 8 | 9 | 9 | 9 | 10 | 10 |
| $r$ | 2 | 2 | 2 | 3 | 2 | 2 | 3 | 4 | 2 | 2 | 3 | 5 | 2 | 4 |
| $k$ | 1 | 2 | 3 | 1 | 4 | 5 | 2 | 1 | 6 | 7 | 3 | 1 | 8 | 2 |
| $c$ | 2 | 6 | 12 | 6 | 20 | 30 | 18 | 12 | 42 | 56 | 36 | 20 | 72 | 36 |

*Zero-step atypicals ($\alpha = 0$): $gl_2(n/1)$ Kac modules $V(k^r, 0^{n-r})$ for $n = 3, 4, ..., 10$*

The restriction to $\alpha = 0$ has been imposed for simplicity as it generates a concretely specified and tractable set of examples. For $\alpha \neq 0$, an infinite series of solutions exists for each value of both $n = 3, 4, 5, ...$ and $r = 1, 2, .., n-1$; the series is indexed by $k = 1, 2, 3, ...$ subject to the additional constraint $\alpha = n - rk + k + r$, which follows directly from (5.27).

## 5.4 Broader Constructions

We consider here two further generalisations of $gl_2(n/1)$. The first arises if the odd generators transform as *densities* under $gl(n)$, rather than vector operators as we have assumed hirtherto. The second is due to the observation in [45] that the satisfaction of certain auxiliary constraints could introduce additional degrees of freedom into the general form of the anticommutator. We review this claim, employing the Diamond Lemma (4.20) to show that the resulting algebra is no longer of PBW-type. Finally, for the $n = 4$ case we present a fermionic realisation of $gl_2(n/1)$ which serves as a concrete instance of both these generalised forms.

**Lemma 5.4.1 (Weighted quadratic superalgebra)**
*Under the modified $gl(n)$ transformation properties with the odd generators being densities of weight $w$,*

$$[E^a{}_b, \overline{Q}^c] = \delta_b{}^c\overline{Q}^a - w\delta_b{}^a\overline{Q}^c\,, \qquad [E^a{}_b, Q_c] = -\delta^a{}_cQ_b + w\delta^a{}_b\overline{Q}^c\,,$$

*the Jacobi constraints on the $gl_2(n/1)$ structure coefficients (as in eq. (5.7), compare* (5.10)*) lead to the following parametric solution*

$$\begin{aligned}\{\overline{Q}^a, Q_b\} &= (E^2)^a{}_b - E^a{}_b\left(\frac{1-2w}{1-nw}\langle E\rangle - \alpha\right) \\ &- \tfrac{1}{2}\delta^a{}_b\left(\langle E^2\rangle - \frac{1-4w+(n+2)w^2}{(1-nw)^2}\langle E\rangle^2 + \left(\frac{n-1+5w-3nw^2}{1-nw} + 2\frac{1-w}{1-nw}\alpha\right)\langle E\rangle\right) + \mathsf{c}\delta^a{}_b\,.\end{aligned} \tag{5.30}$$

**Proof.** The modified commutation relations are (compare (5.6)):

| Symbol | Term | Commutator |
|---|---|---|
| 1(0) | $E^a{}_b$ | $[E^a{}_b, Q_c] = -\delta^a{}_c Q_b + w\delta^a{}_b Q_c$ |
| 0(1) | $\langle E\rangle$ | $[\langle E\rangle, Q_c] = +(nw-1)Q_c$ |
| 2(0) | $(E^2)^a{}_b$ | $[(E^2)^a{}_b, Q_c] = -E^a{}_c Q_b - \delta^a{}_c (QE)_b + 2wE^a{}_b Q_c - w^2\delta^a{}_b Q_c$ |
| 1(1) | $E^a{}_b\langle E\rangle$ | $[E^a{}_b\langle E\rangle, Q_c] = -\delta^a{}_c Q_b\langle E\rangle + (nw-1)E^a{}_b Q_c + w\delta^a{}_b Q_c\langle E\rangle$ |
| 0(2) | $\langle E^2\rangle$ | $[\langle E^2\rangle, Q_c] = -2(QE)_c + nQ_c + 2wQ_c\langle E\rangle - w(2nw-1)Q_c$ |
| $0(1^2)$ | $\langle E\rangle^2$ | $[\langle E\rangle^2, Q_c] = -2Q_c\langle E\rangle + (nw(nw-2)+1)Q_c + 2nwQ_c\langle E\rangle$. |

Following the general method of §5.1 we need only impose the $\overline{111}$ Jacobi identity to the covariant form (5.7) in order to determine (5.30). This simply requires that the double anticommutator $[\{\overline{Q}^a, Q_b\}, Q_c]$ be antisymmetric in $b$ and $c$. We expand and collect the various terms and their coefficients as follows (compare (5.9)):

$$\begin{array}{rlrl} Q_b: & -\mathsf{b}_2 & Q_c: & -w^2 + (n - w(2nw-1))\mathsf{a} \\ & & & \quad +(1+nw(nw-2))\mathsf{c}_1 + w\mathsf{b}_2 + (nw-1)\mathsf{c}_2 \\ E^a{}_c Q_b: & -1 & E^a{}_b Q_c: & 2w + (nw-1)\mathsf{b}_1 \\ (QE)_b: & -1 & (QE)_c: & -2\mathsf{a} \\ Q_b\langle E\rangle: & -\mathsf{b}_1 & Q_c\langle E\rangle: & w\mathsf{b}_1 + 2w\mathsf{a} - 2(1-nw)\mathsf{c}_1. \end{array} \tag{5.31}$$

Antisymmetry in $b$ and $c$ is imposed by equating the coefficients on the left and right for each row of the table above. □

Again recalling the general method employed in §5.1 to derive the final form of the $gl_2(n/1)$ quadratic anticommutator, it is evident from the terms arising in the expansion of the double commutator $[\{\overline{Q}^a, Q_b\}, Q_c]$, namely (5.9) or indeed (5.31) above, that an additional parameter will arise in the final solution if we impose the covariant conditions

$$(E\cdot\overline{Q})^a = (\mathsf{s}\hat{N} + \mathsf{t})\overline{Q}^a \qquad (Q\cdot E)_a = Q_a(\mathsf{s}\hat{N} + \mathsf{t}) \tag{5.32}$$

for constants $\mathsf{s}, \mathsf{t}$. This releases the condition $\mathsf{a} = -\frac{1}{2}$ and the resulting constraints are (compare (5.10))

$$\begin{aligned} &\mathsf{b}_1 = -1 \qquad \mathsf{c}_1 = \frac{1}{2} - \frac{1}{2}(2\mathsf{a}+1)\mathsf{s} \\ &\text{and} \qquad \mathsf{c}_2 = -\alpha - (2\mathsf{a}+1)\mathsf{t} + n\mathsf{a} + \frac{1}{2} - \frac{1}{2}(2\mathsf{a}+1)\mathsf{s} \end{aligned} \tag{5.33}$$

Setting $\alpha \equiv \mathsf{b}_2$, there corresponds a two-parameter family $gl_2(n/1)^{\alpha,\,\mathsf{a},\,\mathsf{c}}$ of quadratic algebras with the anticommutator given by (5.30).

While the additional constraints (5.32) give rise to a more flexible set of defining relations, their introduction as abstract relations has consequences for the PBW theorem. If the form of generalised Jacobi identities (4.19) is altered by the introduction of the new defining relations then Lemma 4.4.7 will no longer hold; that is the fulfillment of the ordinary nested bracket form of the Jacobi identities, as used in the present analysis, is not guaranteed to be the required condition for the existence of a PBW basis.

**Lemma 5.4.2** *The quadratic algebra resulting from the addition of the covariant relations (5.32) to the defining relations of $gl_2(n/1)$ is not a PBW algebra.*

**Proof.** Following the notation of §4.4, we add to the homogeneous relations (4.21) the projection of the additional relations (5.32) onto $X \otimes X$. We express the resulting homogeneous relations in the Gel'fand basis as

$$R = \{ \begin{array}{l} E^i{}_j E^k{}_l - E^k{}_l E^i{}_j, \\ E^i{}_j \overline{Q}^k - \overline{Q}^k E^i{}_j, \\ E^i{}_j Q_k - Q_k E^i{}_j, \\ \overline{Q}^p Q_q + Q_q \overline{Q}^p - (d^p_q)^{kl}_{mn} E^k{}_m E^l{}_n, \\ E^a{}_b Q_a - \mathsf{s}(E^1{}_1 + E^2{}_2 + \cdots + E^n{}_n) Q_b, \\ E^b{}_a \overline{Q}^a - \mathsf{s}(E^1{}_1 + E^2{}_2 + \cdots + E^n{}_n) \overline{Q}^b \quad \}. \end{array} \tag{5.34}$$

The leading terms associated to $R$ are:

$$E^i{}_j E^k{}_l \ (i,j) > (k,l), \quad \overline{Q}^p Q_q, \quad E^n{}_n \overline{Q}^b \text{ and } E^n{}_n Q_b.$$

In other words the set $S_2$ is the same as the ordinary quadratic Lie superalgebra case (see (4.22)) except we must remove the $2n$ elements corresponding to $\overline{01}$ leading monomials $E^n{}_n \overline{Q}^b$ and $E^n{}_n Q_b$, $b = 1,..,n$. To determine the action of $\pi$, we set each expression in (5.34) to zero and rearrange to solve for the leading monomial in each case. (Note: If for a given expression belonging to $R$ one or more of the non-leading terms is the leading term of some other expression then additional substitutions are made until the leading term of each expression is given as a linear sum of non-leading terms.) In particular, we have

$$\pi(E^n{}_n \overline{Q}^b) = \begin{cases} \frac{1}{\mathsf{s}} E^b{}_a \overline{Q}^a - (E^1{}_1 + E^2{}_2 + \cdots + E^{n-1}{}_{n-1}) \overline{Q}^b & (b \neq n) \\ \frac{1}{\mathsf{s}-1} (E^n{}_1 \overline{Q}^1 + E^n{}_2 \overline{Q}^2 + \cdots + E^n{}_{n-1} \overline{Q}^{n-1}) & \\ \qquad - \frac{\mathsf{s}}{\mathsf{s}-1} (E^1{}_1 + E^2{}_2 + \cdots + E^{n-1}{}_{n-1}) \overline{Q}^n & (b = n). \end{cases}$$

We need only find a monomial in $T_3$ which fails the condition (4.20). We take as candidate monomials those which contain as an adjacent pair one of the quadratic terms $E^n{}_n \overline{Q}^b$ or $E^n{}_n Q_b$. Selecting the monomial $E^n{}_n \overline{Q}^b \overline{Q}^c \in T_3$ (assume $c < b \neq n$), we evaluate the left- and right-hand side of (4.20), applying the operators $\pi^{12}$ and $\pi^{23}$ successively until each side is completely ordered and $\pi$ acts trivially,

$$\begin{aligned}
LHS: \quad E^n{}_n\overline{Q}^b\overline{Q}^c &\stackrel{\pi^{12}}{\to} (\tfrac{1}{\mathsf{s}}E^b{}_a\overline{Q}^a - (E^1{}_1 + E^2{}_2 + \cdots + E^{n-1}{}_{n-1})\overline{Q}^b)\overline{Q}^c \\
&\stackrel{\pi^{23}}{\to} \tfrac{1}{\mathsf{s}}E^b{}_a\overline{Q}^a\overline{Q}^c - (E^1{}_1 + E^2{}_2 + \cdots + E^{n-1}{}_{n-1})\overline{Q}^c\overline{Q}^b
\end{aligned}$$

$$\begin{aligned}
RHS: \quad E^n{}_n\overline{Q}^b\overline{Q}^c &\stackrel{\pi^{23}}{\to} E^n{}_n\overline{Q}^c\overline{Q}^b \\
&\stackrel{\pi^{12}}{\to} (\tfrac{1}{\mathsf{s}}E^c{}_a\overline{Q}^a - (E^1{}_1 + E^2{}_2 + \cdots + E^{n-1}{}_{n-1})\overline{Q}^c)\overline{Q}^b \\
&= \tfrac{1}{\mathsf{s}}E^c{}_a\overline{Q}^a\overline{Q}^b - (E^1{}_1 + E^2{}_2 + \cdots + E^{n-1}{}_{n-1})\overline{Q}^c\overline{Q}^b .
\end{aligned}$$

Demanding that $RHS = LHS$ yields the condition,

$$E^b{}_a\overline{Q}^a\overline{Q}^c = E^c{}_a\overline{Q}^a\overline{Q}^b. \tag{5.35}$$

Asides from certain representation-specific cases, such as zero-step modules in which the action of $\overline{Q}^p$ is identically zero, we cannot in the general case expect (5.35) to be satisfied.□

In the absence of the PBW property, the possibility of obtaining a basis for the corresponding non-homogeneous quadratic algebra via Theorem 4.4.3, depends on the homogeneous algebra being Koszul. However, even if the Koszul property is established, it is not evident that there exists a suitable basis for a given homogenous algebra; for example a basis that is generated by an underlying set of ordered pairs and that may be expressed in a factorised form such as raising operators preceding lowering operators.

Finally, adapting slightly the method of proof of Lemma 5.4.1, we combine both of the generalisations presented above to obtain the following result:

**Lemma 5.4.3 (Flexible Quadratic Superalgebra:)**
*The anticommutator bracket structure with weighted generators is relaxed with respect to the Jacobi identities in the presence of additional odd-graded quadratic constraints as above (see (5.32). While coefficient* $\mathsf{b}_1$ *is unchanged from (5.30), namely*

$$\mathsf{b}_1 = -\frac{1-2w}{1-nw}\,,$$

*there remain only 2 constraints on the 4 remaining coefficients* $\mathsf{a}, \mathsf{c}_1, \mathsf{c}_2$ *(with* $\mathsf{b}_2 := \alpha$ *undertermined)*

$$\begin{aligned}
(1-w)\mathsf{b}_1 + \mathsf{s} &= (2w - 2\mathsf{s})\mathsf{a} - 2(1-nw)\mathsf{c}_1\,. \\
(1-w)\mathsf{b}_2 + (\mathsf{t} + w^2) &= (-2\mathsf{t} + n - w(2nw-1))\mathsf{a} + (1 + nw(nw-2))\mathsf{c}_1 + (nw-1)\mathsf{c}_2\,,
\end{aligned}$$

*leading to a 2-parameter solution.* □

**Fermionic oscillator realization**
As a concrete case study of a quadratic $gl_2(n/1)$ superalgebra (with weighted generators as above) we present a construction for the $n = 4$ case using standard fermionic oscillators

(equivalent to an 8-dimensional Clifford algebra) $a_i\,, a^j := (a_j)^\dagger$ , with anticommutation relations

$$\{a_i, a^j\} = \delta_i{}^j\,, \qquad \{a_i, a_j\} = \{a^i, a^j\} = 0\,, \qquad i,j = 1,\cdots,4\,.$$

We define the $gl(n)$ generators

$$E^i{}_j = a^i a_j\,,$$

and odd generators with the use of the totally antisymmetric Levi-Civita tensor,

$$\begin{aligned}\overline{S}^i &= \tfrac{1}{6}\varepsilon^{ik\ell m} a_k a_\ell a_m\,,\\ S_j &= \tfrac{1}{6}\varepsilon_{jpqr} a^k a^\ell a^m\,, \qquad k,\ell,m = 1,\cdots,4\,.\end{aligned}$$

for which the commutation and anticommutation relations are

$$\begin{aligned}[E^i{}_j, E^k{}_\ell] &= \delta_j{}^k E^i{}_\ell - \delta^i{}_\ell E^k{}_j\,,\\ [E^i{}_j, \overline{S}^k] &= \delta_j{}^k \overline{S}^i - \delta_j{}^i \overline{S}^k\,,\\ [E^i{}_j, S_k] &= -\,\delta^i{}_k S_j + \delta^i{}_j \overline{S}^k\,,\\ \text{and}\qquad \{\overline{S}^i, S_j\} &= E^i{}_j(\langle E\rangle - 2) + \delta^i{}_j(-\tfrac{1}{2}\langle E\rangle^2 + \tfrac{3}{2}\langle E\rangle) - \delta^i{}_j\,,\\ \text{with}\qquad \{S_i, S_j\} &= \{\overline{S}^i, \overline{S}^j\} = 0\,.\end{aligned}$$

Given that the construction is framed in the associative algebra of annihilation and creation modes, the Jacobi identities in this case are of course guaranteed. Now the above anticommutator bracket can be written in the general weighted quadratic superalgebra form (Lemma 5.4.1) in different ways, in consequence of the quadratic characteristic relations

$$(E^2)^i{}_j = -E^i{}_j\langle E\rangle + 4E^i{}_j\,, \qquad \langle E^2\rangle = -\langle E\rangle^2 + 4\langle E\rangle$$

satisfied by the generators in this case, by adding a linear combination of these identities and rescaling. In particular it can be checked that

$$\{\overline{S}^i, S_j\} = -\tfrac{3}{4}(E^2)^i{}_j + \tfrac{1}{4}E^i{}_j(\langle E\rangle + 4) + \delta^i{}_j(\tfrac{1}{8}\langle E^2\rangle - \tfrac{3}{8}\langle E\rangle^2 + \langle E\rangle) - \delta^i{}_j$$

is one such equivalent expression. Further, the anticommutator bracket structure with weighted generators (here $w = +1$ ) is relaxed with respect to the Jacobi identities in the presence of additional odd-graded quadratic constraints, of the form given in Lemma 5.4.3 above, with $\mathsf{s} = 4$ , $\mathsf{t} = 0$ in the present realization.

In the present case the remaining Jacobi identity conditions (see Lemma 5.4.3) are, after an overall rescaling of each parameter by $-\frac{3}{4}$ (in addition to $\mathsf{b}_1 = \frac{1}{4}$ ):

$$\mathsf{a} - \mathsf{c}_1 = \tfrac{1}{2}; \qquad -\mathsf{a} + 3\mathsf{c}_1 + \mathsf{c}_2 = -\tfrac{1}{4}\,.$$

These equations are indeed obeyed by the coefficients

$$\mathsf{a} = \tfrac{1}{8}\,; \qquad \mathsf{c}_1 = -\tfrac{3}{8}\,; \qquad \mathsf{c}_2 = 1\,;$$

appearing in the above equivalent form (with $\mathsf{b}_2 \equiv \alpha = -\frac{4}{3}$ ).

# Chapter 6

# QUADRATIC CONFORMAL SUPERSYMMETRY

In this final technical chapter we investigate the possibility of imposing the real form $su_2(2,2/1)$ of the quadratic superalgebra $gl_2(4/1)$ as a non-linear extension of $su(2,2/1)$, the algebra of $N=1$ space-time conformal supersymmetry. This idea was first introduced in [43] where the initial results were derived within the context of oscillator realisations. In this work we undertake to both consolidate and elaborate upon these results within the framework of abstract representation theory utilising the techniques introduced in earlier chapters. We begin by reviewing the structure and properties of $su(2,2/1)$, including a presentation of various bases and the classification, due to Mack, of all lowest weight positive energy unitary representations of the even subalgebra. In particular, we focus on the class of massless representations, where exploiting their so-called degenerate nature, we prove that $su_2(2,2/1)$ satisfies a minimal quadratic identity of degree 2. The main result is to show that the general quadratic anticommutator (5.11) may be brought into correspondence with this minimal identity for precisely the class of massless representations. In other words, the class of massless representations are zero-step modules; the supersymmetry generators are identically zero and the quadratic superalgebra is carried entirely on a single (irreducible) multiplet of the even subalgebra, thus obviating the need for supersymmetric partners.

The content of this chapter closely follows the jointly authored paper [82]; the text and the results are in the largest proportion due to the candidate.

## 6.1 Superconformal Algebra

The algebra of the $N=1$ superconformal spacetime symmetry group is isomorphic to the Lie superalgebra $su(2,2/1)$. The even part is the real form $L_{\bar{0}} \cong u(2,2) \cong su(2,2) + gl(1)$ and the odd part is the $L_0$-module $L_{\bar{1}} \cong \{\bar{1}\} + \{1\}$. In addition to the basis of Gel'fand generators and vector operators, we introduce in this section several alternative bases and establish the mappings between them and other bases appearing in the literature. We recall the classification of all positive energy unitary representations of the even subalgebra as originally given by Mack [56]. In particular, we examine the class of massless representations for which we provide a re-derivation of the massless conditions in our preferred basis. Finally, within the broader classification of unitary representations of $su(2,2)$ as given by Yau [81], we note that all massless representations have the property of being so-called degenerate representations.

As a basis for $su(2,2/1)$ we choose for the even part the Gel'fand generators $E^a{}_b$, $a,b = 1,...,4$ and for the odd part the vector operators $\overline{Q}^c$ and $Q_c$, $c = 1,...,4$. The nonzero commutation relations are

$$[E^a{}_b, E^c{}_d] = \delta^c{}_b E^a{}_d - \delta^a{}_d E^c{}_b. \qquad a,b,c,d = 1,..,n \tag{6.1a}$$

$$[E^a{}_b, \overline{Q}^c] = \delta^c{}_b \overline{Q}^a \quad [E^a{}_b, Q_c] = -\delta^a{}_c Q_b \tag{6.1b}$$

$$\{\overline{Q}^c, Q_d\} = E^c{}_d + \delta^c{}_d Z, \tag{6.1c}$$

where

$$Z \equiv -\langle E \rangle = -E^c{}_c \tag{6.2}$$

is the linear Casimir. In this non-compact case, the requirement of having unitary representations imposes the following hermiticity conditions

$$(E^a{}_b)^\dagger = \eta^b{}_{b'} E^{b'}{}_{a'} \eta^{a'}{}_a, \qquad (Q_a)^\dagger = \eta^a{}_{b'} \overline{Q}^{b'}, \tag{6.3}$$

where $\eta = \mathrm{diag}(-1,-1,1,1)$ [1]. The advantage of this basis is its direct correspondence with the generators of the quadratic superalgebra $gl_2(4/1)$ as well as its straightforward connection with the characteristic identities of $gl(n)$ to be explored in §6.2.

An alternative basis for the even part arises directly from the isomorphism of real Lie algebras $so(4,2) \cong su(2,2)$. We denote by $J_{AB} = -J_{BA}$, $A,B = 0,1,2,3,5,6$ the generators of $so(4,2)$. These together with the additional $gl(1)$ generator $Z$ satisfy the commutation relations

$$\begin{aligned} &[J_{AB}, J_{CD}] = \mathrm{i}(g_{BC}J_{AD} - g_{AC}J_{BD} + g_{AD}J_{BC} - g_{BD}J_{AC}) \\ &[J_{AB}, Z] = 0 \end{aligned} \tag{6.4}$$

where $g = \mathrm{diag}(1,-1,-1,-1,-1,1)$. To facilitate a mapping between this basis for $so(4,2) + gl(1)$ and the previous for $u(2,2)$ we define in the fundamental representation the following basis for $u(2,2)$

$$\begin{aligned} &m_0 = \frac{1}{2}\begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix} \quad n_0 = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix} \quad x_0^+ = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{I} \\ 0 & 0 \end{pmatrix} \quad x_0^- = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ \mathbb{I} & 0 \end{pmatrix} \\ &m_i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix} \quad n_i = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_i \end{pmatrix} \quad x_i^+ = \frac{1}{2}\begin{pmatrix} 0 & \sigma_i \\ 0 & 0 \end{pmatrix} \quad x_i^- = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ \sigma_i & 0 \end{pmatrix} \end{aligned} \tag{6.5}$$

[1] This basis may easily be brought into correspondence with that of [7, 26]. Set $L^a{}_b = E^a{}_b - \frac{1}{4}\delta^a{}_b\langle E \rangle$ and define $T^a{}_b = \eta^a{}_{a'} L^{a'}_{b'} \eta^{b'}{}_b$ satisfying

$$[T^a{}_b, T^c{}_d] = \eta^a{}_d T^c{}_b - \eta^c{}_b T^a{}_d.$$

These generators together with $\overline{Q}^a$, $Q_a$ and $Z$ satisfying $[Z, L^a{}_b] = 0$, $[Z, \overline{Q}^a] = -\overline{Q}^a$ and $[Z, Q_a] = Q_a$ form the required basis.

where $\sigma_i$, $i = 1,2,3$ are the standard Pauli matrices and $\mathbb{I}$ =diag$(1,1)$. The corresponding abstract generators are obtained by expressing each of the concrete generators (6.5) as a linear combination of elementary matrices ${e^a}_b$, then replacing each elementary matrix with ${E^a}_b$. For example,

$$\begin{aligned} m_i &= ({m_i}^{\mathsf{T}})^b{}_a\, {e^a}_b \quad &&\to M_i = ({m_i}^{\mathsf{T}})^b{}_a\, {E^a}_b \\ n_i &= ({n_i}^{\mathsf{T}})^b{}_a\, {e^a}_b \quad &&\to N_i = ({m_i}^{\mathsf{T}})^b{}_a\, {E^a}_b \qquad \text{etc.,} \end{aligned} \tag{6.6}$$

where $^{\mathsf{T}}$ denotes the transpose and the summation occurs over both $a$ and $b$. These satisfy the commutation relations

$$\begin{array}{ll} [M_i, M_j] = \mathrm{i}\varepsilon_{ijk} M_k & [N_i, N_j] = \mathrm{i}\varepsilon_{ijk} N_k \\ [M_0, X_0^+] = \frac{1}{2} X_0^+ & [N_0, X_0^+] = -\frac{1}{2} X_0^+ \\ [M_0, X_i^+] = [M_i, X_0^+] = \frac{1}{2} X_i^+ & [N_0, X_i^+] = [N_i, X_0^+] = -\frac{1}{2} X_i^+ \\ [M_i, X_j^+] = \frac{1}{2} i\varepsilon_{ijk} X_k^+ + \delta_{ij} X_0^+ & [N_i, X_j^+] = \frac{1}{2} i\varepsilon_{ijk} X_k^+ - \delta_{ij} X_0^+ \\ [X_0^+, X_0^-] = \frac{1}{2}(M_0 - N_0) & [X_i^+, X_j^-] = \delta_{ij} \frac{1}{2}(M_0 - N_0) \end{array} \tag{6.7}$$

Following Wybourne [77] (see also Appendix C [71]), we introduce an extended set of Dirac matrices,

$$\gamma_0 = \sigma_3 \otimes \mathbb{I} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \qquad \gamma_i = \mathrm{i}\sigma_2 \otimes \sigma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$$

and define the 6-component vector $\hat{\gamma}_A = (-\gamma_1, -\gamma_2, -\gamma_3, \gamma_5, \gamma_0, -\mathrm{i}\mathbb{I})$ . The operators

$$j_{AB} = \frac{\mathrm{i}}{2} \hat{\gamma}_A \hat{\gamma}_B \qquad A < B, \quad A, B = 1,2,3,4 \equiv 0,5,6. \tag{6.8}$$

together with $z = \frac{1}{2}(\sigma_0 \otimes \mathbb{I})$ provide a four-dimensional matrix representation of $so(4,2) + gl(1)$ satisfying (6.4). Explicit calculation of (6.8) combined with (6.5) generates the following map from the abstract generators of $so(4,2) + gl(1)$ to those of $u(2,2)$:

$$\begin{array}{llll} J_{ij} & = \varepsilon^{ijk}(M_k + N_k) & J_{i5} & = -(M_i - N_i) \\ J_{45} & = X_0^+ - X_0^- & J_{i4} & = \mathrm{i}(X_i^+ + X_i^-) \\ J_{i6} & = -(X_i^+ - X_i^-) & J_{56} & = \mathrm{i}(X_0^+ - X_0^-) \\ J_{46} & = M_0 - N_0 & Z & = M_0 + N_0. \end{array} \tag{6.9}$$

The advantage of the $J_{AB}$ basis is its direct connection with the physical generators of translations, conformal and Lorentz transformations and dilatation, see (6.13). The basis of $M$, $N$, $X^+$, $X^-$ acts as an intermediary between the $J_{AB}$ and the ${E^i}_j$ bases which facilitates computations and the comparability of results.

For $sl(4)$ and its real forms, namely $su(2,2) \cong so(4,2)$, the *lowest weight* of a unitary irreducible representation (when it exists) may be characterised by three real numbers. These numbers may be chosen to be the eigenvalues of the Cartan elements under the module action on the lowest weight vector $|\mu\rangle$ and as such they depend on the choice of basis of the Cartan subalgebra.

The same applies for the Lie superalgebra $su(2,2/1)$ where representations are induced from a single representation of the even subalgebra (see §2.2.4); although in this case an extra label is required owing to the additional Cartan element, the generator of $gl(1)$[2]. For the remainder of this section we focus exclusively on the representation theory of the subalgebra $su(2,2) \subset L_{\overline{0}}$ from which representations of $su(2,2/1)$ may be induced.

We introduce the following Cartan elements for $su(2,2)$ [56]:

$$\begin{aligned} H_0 &\equiv \frac{1}{2}(E^1{}_1 + E^2{}_2 - E^3{}_3 - E^4{}_4) &&= M_0 - N_0 \\ H_1 &\equiv \frac{1}{2}(E^1{}_1 - E^2{}_2) &&= M_3 \\ H_2 &\equiv \frac{1}{2}(E^3{}_3 - E^4{}_4) &&= N_3. \end{aligned} \tag{6.10}$$

The action of these on the lowest weight vector $|\mu\rangle$ is

$$H_0|\mu\rangle = d|\mu\rangle, \qquad H_1|\mu\rangle = -j_1|\mu\rangle, \qquad H_2|\mu\rangle = -j_2|\mu\rangle, \tag{6.11}$$

where $d \in \mathbb{R}$, known as the conformal dimension, labels reducible representations of the Abelian subgroup generated by $H_0$, and $j_1$ and $j_2$, where $2j_1$ and $2j_2$ are non-negative integers, label representations of the compact subgroup $SU(2)\times SU(2)$ generated by $M_1, M_2, (H_1 = M_3)$ and $N_1, N_2, (H_2 = N_3)$. Together these 7 elements generate the *maximal compact subgroup* $SU(2) \times SU(2) \times U(1)$ which plays a key role in the representation theory of $SU(2,2)$; we denote the corresponding subalgebra

$$\mathcal{S} = su(2) \times su(2) \times gl(1).$$

The weight space $H^*$ has a basis of fundamental weights equal in number to the rank $l$, in the present case $l = 3$. The fundamental weights may be projected onto an $(l+1)$-dimensional Euclidean space allowing their expression in terms of an orthonormal basis $\varepsilon_i$, $i = 1, ..., l+1$ satisfying $\varepsilon_i(E^j{}_j) = \delta^i{}_j$ (no sum). Thus we may write $\mu = \sum_{i=1}^4 \mu_i\varepsilon_i$ from which it follows

$$d = \frac{1}{2}(\mu_1 + \mu_2 - \mu_3 - \mu_4), \qquad -j_1 = \frac{1}{2}(\mu_1 - \mu_2) \qquad \text{and} \qquad -j_2 = \frac{1}{2}(\mu_3 - \mu_4). \tag{6.12}$$

The physically relevant classes of unitary irreducible representations of the conformal group are characterised by their energy, mass and spin/helicity. We write the standard physical basis for the conformal algebra as follows[3]:

$$\begin{array}{lll} \text{(Translations)} & P_\mu & = J_{5\mu} + J_{6\mu} \\ \text{(Conformal transformations)} & K_\mu & = J_{6\mu} - J_{5\mu} \\ \text{(Dilations)} & D & = J_{65} \\ \text{(Lorentz transformations)} & M_{\mu\nu} & = J_{\mu\nu}. \end{array} \qquad \mu = 0,1,2,3. \tag{6.13}$$

Mack has shown that there are 5 classes of positive energy, $P_0 \geq 0$, unitary representations of $su(2,2)$, all of which possess a lowest weight $\mu = (d; -j_1, -j_2)$. These are characterised

[2]In the massless case, see (6.14e) , representations of the superconformal algebra comprise only $L_{\overline{0}}$-representations which are themselves massless. This leads to so-called short-multiplets where, irrespective of atypicality, those even submodules which are not massless are decoupled from the induced module, see for example [7][26].

[3]This basis appears in the classic texts [77, 75, 68]. See also [10].

by their Poincaré content, $m$ =mass and $s$ =spin resp. helicity, as follows [56]:

**Trivial**

(1) $d = j_1 = j_2 = 0.$ (6.14a)

**Massive** $m > 0$

(2) $j_1 \neq 0,\ j_2 \neq 0,\ d > j_1 + j_2 + 2 \qquad s = |j_1 - j_2|...j_1 + j_2$ (6.14b)

(3) $j_1 j_2 = 0,\ d > j_1 + j_2 + 1 \qquad s = j_1 + j_2$ (6.14c)

(4) $j_1 \neq 0, j_2 \neq 0,\ d = j_1 + j_2 + 2 \qquad s = j_1 + j_2.$ (6.14d)

**Massless**

(5) $j_1 j_2 = 0,\ d = j_1 + j_2 + 1 \qquad \text{helicity } = j_1 - j_2.$ (6.14e)

The last of these, the class of conformally invariant massless representations, are those on which the operator $P^\mu P_\mu$ acts trivially (starting with this fact we provide in §A.2 an algebraic re-derivation of these massless conditions). The class of positive energy representations may be placed within the broader classification of unitary representations of $su(2, 2)$, see for example [73, 63] and in particular [80, 81]. Unitary representations, each of which comprise an infinite sum of finite-dimensional irreducible representations of $\mathcal{S}$, may be classified as either continuous or discrete representations depending on the allowable values of $j_1, j_2$ and $d$. Within each of these classes exist so-called *degenerate* representations in which distinct representations of $\mathcal{S}$ occurs at most once. Let $|k_1, m_1; k_2, m_2; d'\rangle$, $-k_1 \leq m_1 \leq k_1$, $-k_2 \leq m_2 \leq k_2$ be a (weight) basis for an irreducible $\mathcal{S}$-module satisfying

$$\begin{aligned}
\boldsymbol{M}^2|k_1, m_1; k_2, m_2; d'\rangle &= k_1(k_1 + 1)|k_1, m_1; k_2, m_2; d'\rangle \\
\boldsymbol{N}^2|k_1, m_1; k_2, m_2; d'\rangle &= k_2(k_2 + 1)|k_1, m_1; k_2, m_2; d'\rangle \\
H_l|k_1, m_1; k_2, m_2; d'\rangle &= m_l|k_1, m_1; k_2, m_2; d'\rangle \qquad l = 1, 2 \\
H_0|k_1, m_1; k_2, m_2; d'\rangle &= d'|k_1, m_1; k_2, m_2; d'\rangle.
\end{aligned}$$

A fact that will be crucial to the proof of Lemma 6.2.1 is that for degenerate representations every non-zero weight vector $|k_1, m_1; k_2, m_2; d'\rangle$ occurring in an $\mathcal{S}$-submodule occurs in the $su(2, 2)$ representation with unit multiplicity[80][4]. In the analysis of Yau [80], representations of $\mathcal{S}$ are labeled by $p = k_1 + k_2$, $q = k_1 - k_2$ and $d'$. For the *lowest* (in terms of the extremal weight) of all these representations we have the correspondence $k_1 \to -j_1$ and $k_2 \to -j_2$ denoted $p_0 = -(j_1 + j_2)$ and $q_0 = -(j_1 - j_2)$. Importantly, the class of massless representations are degenerate; they are easily shown to satisfy

$$\begin{aligned}
&p_0 \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, ...\} \\
&q = \pm p_0 \\
&d' = p + 1
\end{aligned}$$

which are the required conditions to belong, in the classification of Yau, to the series of exceptional (most) degenerate discrete representations (see §4 of [81])[5].

[4]For non-degenerate representations an additional (cubic) operator must be constructed in order to form a maximal set of mutually commuting operators and resolve the labelling of states.

[5]For massless representations the condition $d' = k_1 + k_2 + 1$ and $k_1 - k_2 = \mathsf{const}$ reduces the number of state labels from five to just three. This agrees with the characterisation of the most degenerate representations obtained in [73], see also [16] [57].

## 6.2 Minimal identities of $u(2,2)$

Take $L = gl(n)$ in the Gel'fand basis. We set $z = \mathsf{C}_2 = E^i{}_j E^j{}_i$, $V(\varphi) = \{\bar{1}\}$ the fundamental contragredient module with lowest weight $\varphi = -\varepsilon_1$ and $V(\mu)$ an arbitrary lowest weight $L$-module with lowest weight $\mu = \sum_{i=1}^n \mu_i \varepsilon_i$. Following closely the example of §3.2.6, the matrix array (3.21), here denoted by $E$ instead of $A$, takes the form

$$\begin{aligned} E &= -\pi_\varphi(E^j{}_i) \otimes \pi_\mu(E^i{}_j) \\ &= e^i{}_j \otimes \pi_\mu(E^i{}_j). \end{aligned} \tag{6.15}$$

Thus, $E$ is simply the standard matrix array of Gel'fand generators in the representation $\pi_\mu$. We abuse notation denoting by $E^i{}_j$ both the abstract Gel'fand generator and also the $(i,j)$-matrix element of the array $E$. We recall that the $n$-linearly independent Casimir elements generating the centre of $U(gl(n))$ can be written $\mathsf{C_n} = \langle E^n \rangle$ where the matrix powers are defined in the obvious way $(E^k)^a{}_b = (E^{k-1})^a{}_c E^c{}_b$ and $\langle\ \rangle$ denotes the trace.

The characteristic roots of $E$ are (3.52)

$$a_i = \mu_i + i - 1.$$

We saw in §6.1 that the basis $E^a{}_b$ $a,b = 1,..,4$ for $gl(4)$ may equally be viewed as a basis for the non-compact real form $u(2,2)$ subject, in the case of unitary representations, to the Hermiticity conditions (6.3). We can therefore, in the present context, view $E$ as an array of $u(2,2)$ generators and use the results above to determine the corresponding characteristic identities. In contrast to $gl(n)$, the determination of the minimal identity in this non-compact case is more complicated due to the non-unitarity of $V(\varphi)$ and the infinite-dimensionality of the positive energy unitary representations $V(\mu)$.

Using (6.12) we may write without any loss of generality the Cartesian components of an arbitrary lowest weight $\mu$ as

$$\begin{aligned} \mu_1 &= k & \mu_3 &= -d + j_1 - j_2 + k \\ \mu_2 &= 2j_1 + k & \mu_4 &= -d + j_1 + j_2 + k, \end{aligned} \tag{6.16}$$

where $k$ is related to the eigenvalue of the $gl(1)$ generator (6.2)

$$Z = -\langle E \rangle \longrightarrow -\sum \mu_i = 2d - 4j_1 - 4k. \tag{6.17}$$

In terms of these labels the characteristic roots (3.52) are

$$\begin{aligned} a_1 &= k & a_3 &= -d + j_1 - j_2 + k + 2 \\ a_2 &= 2j_1 + k + 1 & a_4 &= -d + j_1 + j_2 + k + 3. \end{aligned} \tag{6.18}$$

We proceed by investigating the existence of a lowest weight vector in each of the subspaces $Y_i$. If $Y_i$ is nonzero then it must, given that $Y$ is the product of two lowest weight modules, contain a lowest weight vector $|\Lambda_i\rangle$ of weight $\mu + \varphi_i$ [51].

**Lemma 6.2.1 (Quadratic polynomial identities of $u(2,2)$)** *Take $\mu$, as given by (6.16), to be the lowest weight of a unitary representation of $u(2,2)$ which under the restriction to*

*$su(2,2)$ satisfies the massless conditions* (6.14e). *For $\varphi$ the* lowest *weight of the fundamental contragredient representation the matrix array $E = \pi_\varphi(E^i{}_k) \otimes \pi_\mu(E^k{}_j)$ satisfies the following quadratic identities:*

$$(E - a_1)(E - a_3) = 0 \qquad (\textit{when } j_1 = 0) \tag{6.19a}$$

$$(E - a_1)(E - a_2) = 0 \qquad (\textit{when } j_2 = 0)\,, \tag{6.19b}$$

*where*

$$a_i = \mu_i + i - 1.$$

**Proof.** $E$ generically satisfies the quartic identity $\prod_{i=1}^4 (E - a_i)$. The proof consists of analysing the decomposition of $Y \equiv V(\varphi) \otimes V(\mu)$ to determine the conditions for which the subspaces $Y_i$ are trivial. For each trivial subspace the degree of the minimal identity satisfied by $E$ is reduced by one where the new identity is obtained from the quartic identity by deletion of the corresponding factors.

For a subspace $Y_i$, $i = 1, 2, 3, 4$ to be non-trivial it must contain a lowest weight vector of weight $\mu + \varphi_i$ where $\varphi_i = -\varepsilon_i$ are the weights of $V(\varphi)$ ordered from lowest to highest. Let $|-\varepsilon_i\rangle$ denote the corresponding weight vectors satisfying $E^a{}_b|-\varepsilon_c\rangle = -\delta^a{}_c|-\varepsilon_b\rangle$. For each $Y_i$ we introduce candidate lowest weight vectors $|\Lambda_i\rangle$ consisting of the most general linear combination of all vectors $|\varphi'\rangle \otimes |\mu'\rangle \in Y$ with weight $\mu + \varphi_i$. Since the massless representations of $su(2,2)$ are degenerate, each weight vector $|\mu'\rangle$ occurs with unit multiplicity and is uniquely characterised by the eigenvalues $m_1$, $m_2$ and $d'$ of $H_1$, $H_2$ and $H_0$ respectively. This significantly reduces the number of available terms when compared with the general non-degenerate case, viz

$$\begin{aligned}
|\Lambda_1\rangle &\equiv |-\varepsilon_1\rangle \otimes |\mu\rangle \\
|\Lambda_2\rangle &\equiv \mathsf{a}_1|-\varepsilon_2\rangle \otimes |\mu\rangle + \mathsf{a}_2(|-\varepsilon_1\rangle \otimes E^1{}_2|\mu\rangle) \\
|\Lambda_3\rangle &\equiv \mathsf{b}_1|-\varepsilon_3\rangle \otimes |\mu\rangle + \mathsf{b}_2(|-\varepsilon_2\rangle \otimes E^2{}_3|\mu\rangle) + \mathsf{b}_3(|-\varepsilon_1\rangle \otimes E^1{}_3|\mu\rangle) \\
|\Lambda_4\rangle &\equiv \mathsf{c}_1|-\varepsilon_4\rangle \otimes |\mu\rangle + \mathsf{c}_2(|-\varepsilon_3\rangle \otimes E^3{}_4|\mu\rangle) \\
&\qquad + \mathsf{c}_3(|-\varepsilon_2\rangle \otimes E^2{}_4|\mu\rangle) + \mathsf{c}_4(|-\varepsilon_1\rangle \otimes E^1{}_4|\mu\rangle).
\end{aligned}$$

To determine the existence of non-trivial values for the constants above we demand that the simple negative root vectors $E^{a+1}{}_a$, $a = 1, 2, 3$ annihilate each of $|\Lambda_i\rangle$. Clearly $E^{a+1}{}_a|\Lambda_1\rangle = 0$, $a = 1, 2, 3$, hence $Y_1$ is always non-trivial. For $|\Lambda_2\rangle$ we have

$$\begin{aligned}
E^2{}_1|\Lambda_2\rangle &= -\mathsf{a}_1|-\varepsilon_1\rangle \otimes |\mu\rangle + \mathsf{a}_2(|-\varepsilon_1\rangle \otimes E^2{}_1 E^1{}_2|\mu\rangle) \\
&= -(\mathsf{a}_1 + 2\mathsf{a}_2 j_1)|-\varepsilon_1\rangle \otimes |\mu\rangle \\
E^3{}_2|\Lambda_2\rangle &= E^4{}_3|\Lambda_2\rangle = 0.
\end{aligned}$$

It follows immediately that $Y_2$ is non-trivial if and only if $j_1 \neq 0$ which implies $j_2 = 0$.

Proceeding similarly for $|\Lambda_3\rangle$,

$$\begin{aligned}
E^2{}_1|\Lambda_3\rangle &= -\mathsf{b}_2|-\varepsilon_1\rangle \otimes E^2{}_3|\mu\rangle + \mathsf{b}_3(|-\varepsilon_1\rangle \otimes E^2{}_1E^1{}_3|\mu\rangle) \\
&= -(\mathsf{b}_2 - \mathsf{b}_3)|-\varepsilon_1\rangle \otimes E^2{}_3|\mu\rangle \\
E^3{}_2|\Lambda_3\rangle &= -\mathsf{b}_1|-\varepsilon_2\rangle \otimes |\mu\rangle + \mathsf{b}_2(|-\varepsilon_2\rangle \otimes E^3{}_2E^2{}_3|\mu\rangle) + \mathsf{b}_3(|-\varepsilon_1\rangle \otimes E^3{}_2E^1{}_3|\mu\rangle) \\
&= -(\mathsf{b}_1 + \mathsf{b}_2(d - j_1 + j_2))|-\varepsilon_2\rangle \otimes |\mu\rangle - \mathsf{b}_3(|-\varepsilon_1\rangle \otimes E^1{}_2|\mu\rangle) \\
E^4{}_3|\Lambda_3\rangle &= 0,
\end{aligned}$$

where for non-trivial $Y_3$, $E^3{}_2|\Lambda_3\rangle = 0$ if and only if $E^1{}_2|\mu\rangle = 0$ which implies $j_1 = 0$. Finally it is straight-forward to show that $Y_4$ is always trivial, in particular

$$\begin{aligned}
E^4{}_3|\Lambda_4\rangle = -\mathsf{c}_1|-\varepsilon_3\rangle \otimes |\mu\rangle &+ \mathsf{c}_2(|-\varepsilon_3\rangle \otimes E^4{}_3E^3{}_4|\mu\rangle) \\
&+ \mathsf{c}_3(|-\varepsilon_2\rangle \otimes E^4{}_3E^2{}_4|\mu\rangle) + \mathsf{c}_4(|-\varepsilon_1\rangle \otimes E^4{}_3E^1{}_4|\mu\rangle) \\
\neq 0.
\end{aligned}$$

Using the above results we have for $j_2 = 0$ and $j_1 = 0$ the decompositions $Y = Y_1 \oplus Y_2$ and $Y = Y_1 \oplus Y_3$ respectively. $\square$

We introduce now the quadratic superalgebra $su_2(2,2/1)^{\alpha,\mathsf{c}}$, a real form of $gl_2(4/1)^{\alpha,\mathsf{c}}$, satisfying the commutation relations (6.1a) and (6.1b) together with (5.11). In light of Lemma 5.1.1, we may view $su_2(2,2/1)^{\alpha,\mathsf{c}}$ for $\mathsf{c} = 0$ as a quadratic deformation of the ordinary superconformal algebra $su(2,2/1)$. We investigate the possiblity of expressing the quadratic right-hand-side of the anticommutator in terms of the minimal identities (6.19). We continue in the Gel'fand basis satifying (2.19) and (5.3).

**Lemma 6.2.2** *The anticommutator for $su_2(2,2/1)^{\alpha,\mathsf{c}}$ may be brought into correspondence with the quadratic identities* (6.19) *for massless representations of the even subalgebra. The coincidence occurs for the parameter value $\alpha = -3$ and for central charge $\mathsf{c} = 0$.*

**Proof.** We begin by noting that for $\alpha = -3$ the anticommutator (5.11) may be expressed in the form

$$\{\overline{Q}^a, Q_b\} = p(E)^a{}_b + \delta^a{}_b\mathsf{Tr}[p(E)].$$

It remains to show that $p(E) = E(E - (\langle E\rangle + 3))$ coincides with the quadratic identities (6.19). Let us start with the $j_1 = 0$ case; we require

$$(E - a_1)(E - a_3) = E(E - (\langle E\rangle + 3))$$

where from (6.18) and (6.17) we have

$$a_1 = k, \quad a_3 = -d - j_2 + k + 2 \quad \text{and} \quad \langle E\rangle = -2d.$$

Setting $a_1 = k = 0$ it follows immediately that $a_3 = \langle E\rangle + 3$ for $d = j_2 + 1$. Similarly for the $j_2 = 0$ case we require

$$(E - a_1)(E - a_2) = E(E - (\langle E\rangle + 3))$$

where from (6.18) and (6.17) we have

$$a_1 = k, \quad a_2 = 2j_1 + k + 1 \quad \text{and} \quad \langle E \rangle = -2d + 4j_1 + 4k.$$

Setting $a_1 = k = 0$ it follows immediately that $a_2 = \langle E \rangle + 3$ for $d = j_1 + 1$. $\square$

We have shown that for the fixed parameter choices $\alpha = -3$ and $\mathsf{c} = 0$ the anticommutator of the quadratic superalgebra $L = su_2(2,2/1)^{\alpha,\mathsf{c}}$ takes the form

$$\{\overline{Q}^a, Q_b\} = E(E - (\langle E \rangle + 3)). \tag{6.20}$$

This is identically zero for $L$-representations induced from representations of even subalgebra $L_0 = u(2,2)$ in which the $su(2,2)$ subalgebra realises conformally invariant massless representations $V_0(\mu) = V_0(j_1, j_2; d)$ where $j_1$, $j_2$ and $d$ satisfy (6.14e). In this scenario the entire $L$-representation is in fact carried on this single $L_0$-module; consider the induced module (2.50)

$$V(\mu) = U(L)T_+.V_0(\mu - 2\rho_1)$$

where $V_0(\mu) = T_+.V_0(\mu - 2\rho_1)$, it follows from $\{L_{+1}, L_{-1}\} = 0$ that $U(L_{-1})T_+ = 0$ and therefore $V(\mu) = V_0(\mu)$. Thus for massless multiplets this extended quadratic supersymmetry remains unbroken while at the same time superpartners do not exist.

This is not the case for $L$-representations induced from positive energy $L_0$-modules for which $m > 0$. The non-trivial commutator introduces the possibility of both typical and atypical representations. We take up discussion of this topic in the concluding chapter where investigation of this and other aspects of the representation theory of quadratic superalgebras must ultimately be left for future work. Finally we direct to the reader to §A.3 where a review is given of the construction of oscillator representations of $su(2,2)$ which connects the present work with the results originally presented in [43].

# Chapter 7

# CONCLUSION

In this thesis we have introduced a class of quadratic deformations of the universal enveloping algebra of Lie superalgebras which we termed quadratic superalgebras. These possess an underlying $\mathbb{Z}_2$-graded vector space comprising an odd and even part; the even part is an ordinary Lie algebra, the odd part is a module of the even part and the anticommutator of two odd elements closes quadratically on the even generators. Due to its non-linear nature, the algebra itself is defined (in complete analogy with the universal enveloping algebra of a Lie algebra) as the tensor algebra of the underlying vector space factored by the ideal generated by the defining (graded commutation) relations. For this class we have proven in chapter 4 a PBW-type theorem which includes the derivation of an explicit ordered basis. We have taken steps toward the development of a representation theory including the investigation of the structure of certain atypical modules and the discovery of zero-step atypicals; these are atypical representations comprising just a single irreducible module of the even subalgebra.

We presented in chapter 5 a concrete class of quadratic superalgebras for which the even part is the Lie algebra $gl(n)$. We have shown that this class, termed $gl_2(n/1)$, constitutes a one-parameter family of algebras which characterise the most general form of a quadratic anticommutator that is consistent with the generalised Jacobi identities. The ordinary Lie superalgebra $sl(n/1)$ is obtained from $gl_2(n/1)$ as a contraction limit. We also identify a large class of finite-dimensional irreducible zero-step $gl_2(n/1)$-modules.

Finally, in chapter 6 we pursue an application of $gl_2(n/1)$ in the context of space-time conformal supersymmetry. For the $n = 4$ case we show that the algebra of spacetime conformal supersymmetry $su(2,2/1)$ is a contraction of the quadratic superalgebra $gl_2(2,2/1)$. Employing the general theory of characteristic identities [35] we prove that the generators of the even subalgebra $u(2,2) \cong su(2,2)+gl(1)$ satisfy quadratic minimal identities for precisely the class of lowest weight conformally invariant massless representations (in the classification of Mack [56]). Furthermore, for a specific parameter choice of $gl_2(2,2/1) = gl_2(2,2/1)^{\alpha,\mathsf{c}}$ these quadratic identities are brought into correspondence with the anticommutator. As a consequence, when postulated as an extended symmetry principle underlying the particle spectrum, the quadratic superconformal algebra provides a mechanism for all standard model fields (seen as massless conformal irreducible representations at unification scales) to be identified with zero-step modules; that is, with no super-partner fields.

As discussed in the introduction, the class of algebras introduced in this thesis may be viewed as part of a larger trend in theoretical physics to investigate non-linear algebraic structures. While in the context of particle classification and the quest for a unified model this may be motivated by the imperative to extend beyond the standard model, the merits of studying non-linear symmetry algebras more generally stem from the fact that there is no physical reason to restrict candidate symmetry groups to those which are linear. This view is promoted by de Boer et al. [20] where simple examples of non-linear symmetries (in this case concerning finite $\mathcal{W}$-algebras) are shown to arise in elementary systems.

The general framework of quadratic algebras, reviewed in §4.4, constitutes the broadest possible class of non-linear algebras for which the defining relations are of maximal degree two [61]. Within this class there exists a subclass of *homogeneous* quadratic PBW algebras for which an explicit PBW basis may be derived given a fixed ordering of the underlying vector space. Associated with each homogeneous PBW algebra is a family of *nonhomogeneous* algebras, which may also be termed PBW algebras, subject to the fulfillment of the generalised Jacobi identities [12]. These nonhomogeneous PBW algebras share the same basis as their homogeneous counterparts.

As we have shown in chapter 5, the class of quadratic superalgebras introduced in this thesis belong to the class of quadratic PBW algebras. This provides a suite of useful tools, such as the diamond lemma and of course the generalised PBW theorem, which permit the validity of various modifications to the original defining relations to be examined (see §5.4). Despite the utility of the PBW property in enabling the concrete investigation of irreducible representations (see for example §4.3, §5.2 and §6.2), it is not sufficient to guarantee the existence of any valid coproduct. This stands in contrast to quantum groups where a coproduct is guaranteed due to the underlying structure of a Hopf algebra, but for which the PBW property must be established independently.

As such, the development of the representation theory of quadratic superalgebras remains incomplete and must be left as a topic for further work. A possible direction for this work is to investigate the striking similarity between the quadratic superalgebra $gl_2(n/1)$ and certain examples of finite $\mathcal{W}$-algebras. The latter are reviewed in [21] and are derived by applying a certain (Hamiltonian) reduction formalism on the canonical linear Poisson algebra associated with any Lie algebra. As an illustration, we consider the finite $\mathcal{W}$-algebra associated with the Lie algebra $sl(4)$ characterised by the $sl(2)$ embedding, $\mathbf{4} \to \mathbf{2}_0 + \mathbf{1}_1 + \mathbf{1}_{-1}$. This algebra possesses an undeformed subalgebra $sl(2) + gl(1)$ with generators $E$, $F$, $H$ and $U$, together with a pair of spin $\frac{1}{2}$ representations $\overline{Q}^a$, $Q_a$, $a = 1, 2$ satisfying

$$[U, \overline{Q}^a] = \hbar \overline{Q}^a \qquad \text{and} \qquad [U, Q_a] = -\hbar Q_a.$$

Following [45] we define

$$\begin{pmatrix} E^1{}_1 & E^1{}_2 \\ E^2{}_1 & E^2{}_2 \end{pmatrix} = \begin{pmatrix} U + H & F \\ E & U - H \end{pmatrix}$$

and the quadratic closure relations (A.4) of [21] read

$$[\overline{Q}^a, Q_b] = \mathsf{a}(E^2)^a{}_b + \mathsf{b}E^a{}_b + \mathsf{c}\delta^a{}_b\mathbb{I}.$$

While the above is not a $\mathbb{Z}_2$-graded algebra, it shares significant structural similarities with $gl_2(n/1)$ (compare §5.1). One might investigate, for example, the possibility of certain classes of quadratic superalgebras being deformations of either finite $\mathcal{W}$-algebras or their $\mathbb{Z}_2$-graded counterparts finite $\mathcal{W}$-superalgebras [13, 14, 83, 1].

A significant and motivating context in which quadratic superalgebras arise is the study of gauge invariant fields in Hamiltonian lattice QCD [44]. The observable algebra (see discussion and (1.2) in the introduction) comprises a quadratic superalgebra for which the even part is $gl(n)$ and the odd part, expressed in partition notation as a representation of the even part, is $\{3\}+\{\overline{3}\}$. This example closely resembles the oscillator model presented at the end of §5.4; in fact it was shown in [44] that the observable algebra of Hamiltonian lattice QCD may, with further analysis, be reduced to the latter which is isomorphic to $gl_2(n/1)$. In this lattice-derived context, the possibility of 'pasting' together two identical lattices together with the corresponding analysis of boundary conditions and resultant algebraic structure, may provide insight into the formulation of a co-product for the associated quadratic superalgebra.

In relation to the superconformal application of Chapter 6, we consider some directions for further work. These include the investigation of no-go theorems and their possible extension (weakening) to include non-linear symmetries in analogy with the weakening of assumptions that permitted the Haag−Lopuszański−Sohnius theorem to generalise the original Coleman-Mandula theorem (see for example [54]). This could provide a set of constraints to guide the development of particle models that go beyond local relativistic quantum field theory. It would be of interest to develop physical models based on quadratic supersymmetry wherein the physical significance of free parameters belonging to the algebra could be investigated, in particular their behavior in the limit of relevant (unification-scale) energy thresholds.

As concerns the representation theory of $gl_2(2,2/1)$, the massive ($m > 0$) unitary irreducible representations need to be investigated to identify those which are atypical and to expose their structure expressed as a sum of irreducible modules of the even subalgebra. A method for doing this might be the development of an expression of the atypicality conditions, such as those expressed for Lie superalgebras, in terms of the inner product on $H^*$ (see (2.48)). Given the quadratic exchange relations between odd elements, it is unlikely that action of $T_+T_-$ on elements of $V_0(\lambda)$ would be expressible in a succinct and factorised form; it would none the less be of interest to discover a general rule for atypicality conditions that can be determined via the weight space.

While the first part of this thesis focused on mathematical aspects of the quadratic superalgebras, the latter part, rather than extend the abstract formulation, concerned the physical application of one specific algebra, $gl_2(4/1)$. As such, many aspects of the mathematical theory are left for further work; these include: a classification theory and the identification of root systems; the Casimir invariants; alternative (canonical) bases such as a Chevalley-type basis; and, as discussed above, the existence of a co-product. Concerning classification, the introduction of type I' and type II' algebras, see Definitions 4.1.2 and 4.1.3, was made in analogy with Lie superalgebras and is by no means expected to be an exhaustive typology; there may exist balanced or unbalanced types beyond these two categories which bear no direct relationship, such as a contraction limit, to a specific Lie superalgebra. Finally, going beyond the scope of this thesis, certain polynomial generalisations of quadratic Lie superalgebras were explored in [45]. It would be very interesting to see if the tools and techniques applied in this thesis, such as the derivation of a PBW-basis and the existence of zero-step modules, could be adapted to these higher-degree polynomial algebras, possibly requiring the satisfaction of higher-order Jacobi-type identities.

## List of Symbols

| | |
|---|---|
| $\chi$ | The central character , p.19 |
| $\Delta$ | The set of roots belonging to $L$ , p.11 |
| $\Delta^+$ | The set of positive roots belonging to $L$ , p.11 |
| $\Delta^-$ | The set of negative roots belonging to $L$ , p.11 |
| $\delta^i{}_j$ | Kronecker delta , p.6 |
| $\langle\,,\,\rangle$ | The Killing Form on $L$ , p.9 |
| $\langle\;,\;\rangle$ | The inner product on $H^*$ , p.11 |
| $\Pi$ | The simple root system , p.11 |
| $\rho$ | The half-sum of positive roots for a LA , p.20 |
| $\rho_1$ | half sum of positive odd roots , p.31 |
| $\varepsilon_i$ | Orthonormal basis for $H^*_{\mathbb{R}}$ (Classical Lie Algebras) , p.14 |
| $\wedge$ | The exterior (antisymmetric) product , p.20 |
| $\{\,,\,\}$ | The inner product on $H^*_{\mathbb{R}}$ , p.14 |
| $a_i$ | Characteristic Root , p.44 |
| $A_{ij}$ | The Cartan Matrix , p.12 |
| $c$ | The universal (quadratic) Casimir , p.19 |
| $g_{ij}$ | Metric Tensor , p.10 |
| $H$ | Cartan Subalgebra |
| $H^*_{\mathbb{R}}$ | The real restriction of $H^*$ , p.13 |
| $S(X)$ | The symmetric algebra over $X$ , p.16 |
| $T(X)$ | The tensor algebra over $X$ , p.16 |
| $T_\pm$ | Canonical elements of $U(L_{\pm 1})$ , p.31 |
| $U(L)$ | The universal enveloping algebra , p.16 |
| $V(\varphi)$ | Reference Representation , p.44 |
| $Z$ | The centre of $U(L)$ , (2.20) |
| $\mathsf{C}_k$ | Casimir Invariant of degree $k$ , p.20 |

| | |
|---|---|
| h.w. | Highest Weight |
| l.w. | Lowest Weight |
| Unirrep | Unitary Irreducible Representation |

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# Appendix A

The first section (§A.1) of this chapter contains a result due to Jarvis [82] concerning an isomorphism between certain instances of $gl_2(n/1)^{\alpha,\mathsf{c}}$ due to an additive shift in the linear Casimir. The remaining sections also appear in the jointly authored paper [82] and are due to the candidate. They contain a new (algebraic) derivation of Mack's conditions for masslessness in the context of positive energy unitary representation of the conformal group (§A.2) and a brief overview of the construction of oscillator representation for $su(2,2)$ (§A.3).

## A.1 Shifting Lemma

Using the definitions and notation of chapter 5 we consider the effect of an overall shift $\lambda'_i = \lambda_i + \frac{\eta}{n}$ of an arbitrary representation $V_0(\lambda)$ of the even subalgebra of $gl_2(n/1)^{\alpha,\mathsf{c}}$. These are rational (tensor density) representations equivalent to the tensor product of a finite-dimensional tensor representation with highest weight $\lambda$ with the one-dimensional determinant representation. The weights $\lambda'_i - \lambda'_{i+1} = \lambda'_i - \lambda'_{i+1}$ of the $sl(n)$-subalgebra are preserved however the eigenvalue of the $gl(1)$ generator shifts by an additive constant, viz

$$\langle E' \rangle = \langle E \rangle + \eta \mathbb{I}.$$

Equivalent to this shift is an additive redefinition of the even generators together with appropriately adjusted values for each of the parameters $\alpha$ and $\mathsf{c}$. The following result is obtained via explicit computation:

**Lemma A.1.1 (Shifting isomorphism)**
*Let $gl_2(n/1)^{\alpha,\mathsf{c}}$ be the quadratic superalgebra with bracket relations (5.11)*

$$\{\overline{Q}^a, Q_b\} = (E^2)^a{}_b - E^a{}_b\big(\langle E \rangle - \alpha\big) - \tfrac{1}{2}\delta^a{}_b\big(\langle E^2 \rangle - \langle E \rangle^2 + (n-1+2\alpha)\langle E \rangle\big) + \mathsf{c}\delta^a{}_b\,.$$

*Define*

$$\sigma\big(E^a{}_b\big) = E^a{}_b - \frac{\lambda}{n}\delta^a{}_b\mathbb{I}\,, \qquad \sigma\big(\overline{Q}^a\big) = \overline{Q}^a\,, \qquad \sigma\big(Q_b\big) = Q_b\,,$$

*Then we have an isomorphism of algebras $gl_2(n/1)^{\alpha,\mathsf{c}} \cong gl_2(n/1)^{\sigma(\alpha),\sigma(\mathsf{c})}$ with*

$$\sigma(\alpha) = \alpha - (n-2)\frac{\lambda}{n}\,, \qquad \sigma(\mathsf{c}) = c - (n-1)\frac{\lambda}{n}(\alpha - \tfrac{1}{2}(n-2)\frac{\lambda}{n} + \tfrac{1}{2}n)\,.$$

*In particular for the choice $\lambda/n = \alpha/(n-2)$, we have*

$$gl_2(n/1)^{\alpha,\mathsf{c}} \cong gl_2(n/1)^{0,\bar{\mathsf{c}}}\,, \qquad \bar{\mathsf{c}} = \mathsf{c} - \tfrac{1}{2}\frac{(n-1)}{(n-2)}\alpha(\alpha+n)\,.$$

□

## A.2 Derivation of Massless Conditions

We provide in the first part of this appendix an algebraic re-derivation of the massless conditions (6.14e) starting from the assumption $P^\mu P_\mu = 0$ on massless multiplets. In the second part we review various oscillator constructions for $su(2,2)$ and $so(4,2)$.

Let $\mathcal{M}$ be a massless representation, $u \in U(so(4,2))$ and $|\psi\rangle \in \mathcal{M}$. Since $\mathcal{M}$ is invariant under the action of $so(4,2)$ it follows that

$$[u, P^\mu P_\mu]|\psi\rangle = 0. \tag{A.1}$$

Following [10] we introduce the $so(4,2)$ tensor operators

$$W_{AB} = J_{AC}J^C{}_B + J_{BC}J^C{}_A + \frac{1}{3}g_{AB}J_{CD}J^{CD} \tag{A.2}$$

which generate an 20-dimensional invariant subalgebra $\mathcal{W} \subset U$. Using (6.13) and (A.2) we have

$$\begin{aligned} P^\mu P_\mu &= g^{\mu\nu}(J_{5\mu} + J_{6\mu})(J_{5\nu} + J_{6\nu}) \\ &= -\frac{1}{2}W_{55} - \frac{1}{2}W_{66} - W_{56} \end{aligned}$$

which together with (A.1) leads to the following result.

**Theorem A.2.1 (Massless Conditions ([10], Theorem 3.1))** *A module $\mathcal{M}$ is a conformally invariant massless representation if and only if*

$$W_{AB}|\psi\rangle = 0, \qquad A, B = 0, 1, 2, 3, 5, 6. \tag{A.3}$$

We now prove the equivalence of the massless conditions (6.14e) and (A.3). The set of 20 linearly independent equations (A.3) may be rewritten, taking appropriate linear combinations, in a more convenient form expressed in terms of the physical generators (6.13). This new set naturally includes

$$P^\mu P_\mu|\psi\rangle = 0 \tag{A.4}$$

and we shall require one other, namely (Eq 3.26 [10])

$$K_\mu P^\mu|\psi\rangle = \left(M_{\mu\nu}M^{\mu\nu} - 4iD + 4D^2\right)|\psi\rangle. \tag{A.5}$$

Let $|\psi_\mu\rangle$ be the lowest weight vector of $\mathcal{M}$. We can establish a constraint condition by demanding that the zero-weight contributions of (A.4) and (A.5) hold in the case $|\psi\rangle = |\psi_\mu\rangle$.

Using (6.13)(6.9) we obtain

$$\begin{array}{ll} P_0 = H_0 + (X_0^+ - X_0^-) & K_0 = H_0 - (X_0^+ - X_0^-) \\ \boldsymbol{P} = (\boldsymbol{M} - \boldsymbol{N}) + (\boldsymbol{X}^+ - \boldsymbol{X}^-) & \boldsymbol{K} = (\boldsymbol{M} - \boldsymbol{N}) - (\boldsymbol{X}^+ - \boldsymbol{X}^-) \\ D = -i(X_0^+ - X_0^-) & D^2 = -(X_0^+ - X_0^-)^2. \end{array} \tag{A.6}$$

$\boldsymbol{M}^2$ and $\boldsymbol{N}^2$ are the quadratic Casimir for each of the compact $SU(2)$ subgroups respectively. Their eigenvalues are

$$\boldsymbol{M}^2|\psi\rangle = j_1(j_1+1)|\psi\rangle \qquad \boldsymbol{N}^2|\psi\rangle = j_2(j_2+1)|\psi\rangle.$$

Using the commutation relations (6.7) we evaluate the action

$$\begin{aligned} \left[(P_0)^2\right]_0|\psi_\mu\rangle &= \left((H_0)^2 - X_0^+X_0^- - X_0^-X_0^+\right)|\psi_\mu\rangle \\ &= \left((H_0)^2 + \tfrac{1}{2}H_0\right)|\psi_\mu\rangle \\ &= d(d + \tfrac{1}{2})|\psi_\mu\rangle \end{aligned}$$

and similarly

$$\begin{aligned}\left[\boldsymbol{P}^2\right]_0|\psi_\mu\rangle &= \left(\boldsymbol{M}^2 - 2\boldsymbol{M}\cdot\boldsymbol{N} + \boldsymbol{N}^2 + \tfrac{3}{2}H_0\right)|\psi_\mu\rangle \\ &= (j_1(j_1+1) + j_2(j_2+1) - 2j_1j_2 - \tfrac{3}{2}d)|\psi_\mu\rangle\end{aligned}$$

where $[\quad]_0$ denotes the zero-weight terms of the relevant expression. The requirement

$$[P^\mu P_\mu]_0|\psi_\mu\rangle = \left[(P_0)^2 - \boldsymbol{P}^2\right]_0|\psi_\mu\rangle = 0$$

leads to the condition

$$d(d-1) - j_1(j_1+1) - j_2(j_2+1) + 2j_1j_2 = 0.$$

Proceeding similarly,

$$\begin{aligned}[M_{\mu\nu}M^{\mu\nu}]_0|\psi_\mu\rangle &= \left[2\left(-(M_{10})^2 - (M_{20})^2 - (M_{30})^2 + (M_{12})^2 + (M_{23})^2 + (M_{31})^2\right)\right]_0|\psi_\mu\rangle \\ &= \left(-3H_0 + 2\boldsymbol{M}^2 + 4\boldsymbol{M}\boldsymbol{N} + 2\boldsymbol{N}^2\right)|\psi_\mu\rangle \\ &= (2j_1(j_1+1) + 2j_2(j_2+1) + 4j_1j_2 - 3d)|\psi_\mu\rangle\end{aligned}$$

$$\begin{aligned}[K_0P_0]_0|\psi_\mu\rangle &= \left((H_0)^2 - \tfrac{1}{2}H_0\right)|\psi_\mu\rangle \\ [\boldsymbol{K}.\boldsymbol{P}]_0|\psi_\mu\rangle &= \left(-\boldsymbol{M}^2 + 2\boldsymbol{M}\boldsymbol{N} - \boldsymbol{N}^2 + \tfrac{3}{2}H_0\right)|\psi_\mu\rangle \\ \Rightarrow [K_\mu P^\mu]_0|\psi_\mu\rangle &= (d(d-2) + j_1(j_1+1) + j_2(j_2+1) - 2j_1j_2)|\psi_\mu\rangle.\end{aligned}$$

and

$$\left[D^2\right]_0|\psi_\mu\rangle = -\tfrac{1}{2}H_0|\psi_\mu\rangle = -\tfrac{1}{2}d|\psi_\mu\rangle.$$

Employing all of the results above and subtracting (A.4) from (A.5), we have

$$\begin{aligned}&[K^\mu P_\mu - P^\mu P_\mu]_0|\psi_\mu\rangle = \left[M_{\mu\nu}M^{\mu\nu} - 4iD + 4D^2\right]_0|\psi_\mu\rangle \\ &\Rightarrow 2j_1(j_1+1) + 2j_2(j_2+1) - 4j_1j_2 - d = 2j_1(j_1+1) + 2j_2(j_2+1) + 4j_1j_2 - d \\ &\Rightarrow j_1j_2 = 0.\end{aligned}$$

Substituting $j_1j_2 = 0$ into (A.4) gives two possible cases:

$$\begin{array}{llcll}(j_2=0) & d(d-1) = j_1(j_1+1) & \Rightarrow & d = j_1+1 & \quad\text{or} \\ (j_1=0) & d(d-1) = j_2(j_2+1) & \Rightarrow & d = j_2+1, &\end{array}$$

which together imply

$$d = j_1 + j_2 + 1 \qquad \text{where} \qquad j_1j_2 = 0.$$

Thus (6.14e) are necessary conditions for (A.3). We now show that they are sufficient.

Let $W^*$ be the highest weight of the $\mathcal{W}$. The conditions (A.3) will hold if

$$W^*|\psi_\mu\rangle = 0. \tag{A.7}$$

For on any state $|\psi\rangle = u_+|\psi_\mu\rangle$, $u_+ \in U_+$, we have

$$W^*|\psi\rangle = W^*u_+|\psi_\mu\rangle = \left(u_+W^* + [W^*, u^+]\right)|\psi_\mu\rangle = 0$$

since $[W^*, u_+] = 0\ \forall\ u_+ \in U_+$. There exists, due to irreducibility, $u_{AB} \in U_-$ such that $W_{AB} = [u_{AB}, W^*]$, and it follows that

$$\begin{aligned}W_{AB}|\psi\rangle &= W_{AB}(u_+|\psi_\mu\rangle) \\ &= u_{AB}(W^*u_+|\psi_\mu\rangle) - W^*(u_{AB}u_+|\psi_\mu\rangle) \\ &= 0 - W^*(u_+u_{AB} + [u_{AB}, u_+])|\psi_\mu\rangle \\ &= 0\end{aligned}$$

since $W^*[u_{AB}, u_+]|\psi_\mu\rangle = 0$ for either $[u_{AB}, u_+] \in U_+ \oplus U_0$ or $[u_{AB}, u_+] \in U_-$.

$\mathcal{W} \simeq \mathbf{20}$ transforms as an $so(4,2)$ tensor representation of symmetry type $\{\overline{1^2}; 1^2\}$ with corresponding highest weight $\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4$. Identifying $\mathbf{15}$ as the adjoint representation we have

$$\mathbf{15} \cdot \mathbf{15} \simeq \mathbf{84} + \mathbf{20} + \mathbf{15} + \mathbf{1}.$$

Accordingly we may write a basis for $\mathcal{W}$ that is quadratic in the Gel'fand generators taking the form

$$W^{ij}_{kl} = E^i{}_k E^j{}_l - E^j{}_k E^i{}_l - E^i{}_l E^j{}_k + E^j{}_l E^i{}_k + (\mathit{diagonal\,terms}) - (\mathit{traces})$$

In this basis, we have (up to an overall normalisation)

$$W^* = W^{12}_{34} = E^1{}_3 E^2{}_4 - E^2{}_3 E^1{}_4.$$

Finally we demand,

$$\begin{aligned}
0 &= \langle\psi_\mu|(W^*)^\dagger W^*|\psi_\mu\rangle \\
&= \langle\psi_\mu|(E^4{}_2 E^3{}_1 - E^4{}_1 E^3{}_2)(E^1{}_3 E^2{}_4 - E^2{}_3 E^1{}_4)|\psi_\mu\rangle \\
&= \langle\psi_\mu|(E^3{}_3 - E^1{}_1)(E^4{}_4 - E^2{}_2) + 2(E^3{}_3 - E^2{}_2) + (E^3{}_3 - E^2{}_2)(E^4{}_4 - E^1{}_1)|\psi_\mu\rangle \\
&= 2\langle\psi_\mu|(H_0)^2 - (H_1)^2 - (H_2)^2 - H_0 + H_1 + H_2|\psi_\mu\rangle \\
&= 2\langle\psi_\mu|(d^2 - {j_1}^2 - {j_2}^2 - d - j_1 - j_2)|\psi_\mu\rangle \\
&= 2\langle\psi_\mu|(d - j_1 - j_2 - 1)(d + j_1 + j_2)|\psi_\mu\rangle
\end{aligned}$$

for which the conditions (6.14e) are sufficient. □

## A.3 Oscillator Representations of $su(2,2)$

We review here the construction of oscillator models for $L = u(2,2) \cong so(4,2)$. These can viewed as a special case of a general approach that applies to Lie algebras more broadly, see for example [73] and for Lie superalgebras [2].

We begin with the introduction of the following four-component operator-valued objects

$$\varphi_\pm = \begin{pmatrix} \pm a_1 \\ \pm a_2 \\ b_1^\dagger \\ b_2^\dagger \end{pmatrix} \qquad \tilde{\varphi}_\pm = \varphi_\pm^\dagger \eta_\pm = \left(a_1^\dagger, a_2^\dagger, \mp b_1, \mp b_2\right) \tag{A.8}$$

where $\eta_\pm = \pm\gamma_0 = \pm diag(1,1,-1,-1)$ and the components are the bosonic creation and annihilation operators satisfying the usual relations

$$[\varphi_i, \tilde{\varphi}^j] = \delta^i{}_j \qquad\qquad [\varphi_i, \varphi_j] = [\tilde{\varphi}^i, \tilde{\varphi}^j] = 0.$$

Now for $X \in L$ and a (four-dimensional) representation $\pi : X \mapsto \pi(X)$ there corresponds two inequivalent oscillator realisations, termed $\mathcal{L}_\pm$,

$$X_+ = \tilde{\varphi}_+\, \pi(X)\, \varphi_+ \qquad \text{and} \qquad X_- = \tilde{\varphi}_-\, \pi(X)\, \varphi_-,$$

where the hermiticity conditions (6.3) are satisfied as a result of (A.8). Take for example the Gel'fand generators $E^i{}_j$, $i,j = 1,2,3,4$ as a basis for $su(2,2)$ satisfying (6.1a). In the fundamental representation each $E^i{}_j$ may be mapped to the elementary matrix $e^i{}_j$. The resulting $\mathcal{L}_+$ realisation takes the form

$$E^i{}_j = \tilde{\varphi}^l \left(e^i{}_j\right)^k{}_l \varphi_k = \tilde{\varphi}^i \varphi_j.$$

In the case of $so(4,2)$ take as a basis $J_{AB} = -J_{AB}$, $A, B = 0,1,2,3,5,6$ satisfying (6.4). Using the defining representation $j_{AB}$ expressed in terms of the gamma matrices, see (6.8), the corresponding oscillator realisations are

$$J_{AB} = \tilde{\varphi}_+ \, j_{AB} \, \varphi_+ \qquad \text{and} \qquad J_{AB} = \tilde{\varphi}_- \, j_{AB} \, \varphi_-,$$

which appear in this or equivalent forms in the literature, see for example [57] and references therein. In particular the explicit evaluation of the $\mathcal{L}_-$ case appears in [77] and [3], viz

$$\begin{array}{llll} J_{ij} & = \frac{1}{2}(a^\dagger \sigma_k a + b^\dagger \sigma_k b) & J_{56} & = \frac{1}{2}(a^\dagger C b^\dagger + aCb) \\ J_{i5} & = -\frac{1}{2}(a^\dagger \sigma_i a - b^\dagger \sigma_i b) & J_{50} & = -\frac{1}{2} i (a^\dagger C b^\dagger - aCb) \\ J_{i0} & = -\frac{1}{2}(a^\dagger \sigma_i C b^\dagger - aC\sigma_i b) & J_{06} & = \frac{1}{2}(a^\dagger a + b^\dagger b + 2) \\ J_{i6} & = -\frac{1}{2} i (a^\dagger \sigma_i C b^\dagger + aC\sigma_i b), & & \end{array} \tag{A.9}$$

where $C$ is the antisymmetric matrix

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Finally Mack and Todorov [57], see also [53], have shown that all oscillator representations of $su(2,2)$ are massless representations belonging the most degenerate series of unitary representations. See references above for a discussion of the decomposition of the Fock space into its irreducible parts and the irreducibility of oscillator representations under restriction to the Poincaré subgroup.

## A.4 Generalised Tensor Projection Operators

For completeness, we provide here the definition of the tensor projection operators, derived from the characteristic identity, for infinite-dimensional representations $V_0(\mu)$. We use the notation introduced in §3.2 (see also §5.2). This case generalises the definition for finite-dimensional representations (compare (5.13)), and takes the following form [35]:

$$P[s] \equiv H[s]Q[s],$$

for which

$$Q[s] = q_s(E) \qquad \text{and} \qquad H[s] = \sum_{t=0}^{m_t - 1} \xi_{s,t}(E - a_s)^t$$

with

$$q_s(x) = \prod_{t \neq s} \left( \frac{(x - a_t)}{(a_s - a_t)} \right)^{m_t}$$

and the coefficients $\xi_{s,t}$ given by

$$\xi_{s,t} = \frac{1}{t} \left[ \frac{1}{q_i(x)} \right]^{(t)} \Bigg|_{x = a_i}$$

where $f^{(t)}(x)$ denotes the $t^{th}$ derivative of $f(x)$ and $f^{(0)}(x) = f(x)$.

# INDEX